K-Theory of C^*-algebras

An introduction

$$\begin{array}{ccccc}
K_0(J) & \xrightarrow{j_*} & K_0(A) & \xrightarrow{\rho_*} & K_0(A/J) \\
\uparrow \partial & & & & \downarrow \tilde{\partial} \\
K_1(A/J) & \xleftarrow[\rho_*]{} & K_1(A) & \xleftarrow[j_*]{} & K_1(J)
\end{array}$$

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Preface

From the physicists perspective, C^*-algebras are well motivated to be studied. In quantum mechanics, observables are self-adjoint operators $H \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. This requires the concept of hermitian adjoint $A \mapsto A^\dagger$. The hermition adjoint is an antilinear ($(A + \beta B)^\dagger = A^\dagger + \overline{\beta} B^\dagger$) anti-involution ($(AB)^\dagger = B^\dagger A^\dagger$ and $(A^\dagger)^\dagger = A$). C^*-algebras are Banach algebras with an antilinear anti-involution $*$, together with some continuity assumptions about the norm. This additional structure leads to strong results. For example, every $*$-morphism is norm decreasing, and if it is also injective, it even is always an isometry (see theorem 2.6.10).

K-theory describes a sequence of functors K_n, from the category of (local) C^*-algebras to abelian groups. In fact, for complex K-theory, only K_0 and K_1 are of importance. A central result, the Bott-periodicity, states, that $K_0 \cong K_{2n}$ and $K_1 \cong K_{2n+1}$ for all $n \in \mathbb{N}_0$. The K-groups can be used to analyze and categorize different C^*-algebras.

These notes follow [All17] very closely, to the point, where it is but a translation in some parts. On the other hand, these notes were created while I taught myself K-theory of C^*-algebras, and contain also setps, that may be considered as trivial. So the notes are intended for the novice, rather than the expert.

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Contents

1 Functional analysis for C^*-algebras

This chapter is intended to give an overview of methods from functional analysis, needed for C^*-algebras and is mostly based on [Wer11, chapter VIII(1-3) and IX(2)] with additions from [All17] and [Con97]. In this chapter we introduce some basic concepts of locally convex spaces, weak topologies, Banach algebras and the Stone-Weierstrass theorem. Since this chapter is only a recap of what is needed for C^*-algebras, most of the claims will not be proven here.

1.1 Locally convex spaces

We recall that a **semi norm** on a vector space X is a map $p\colon X \to \mathbb{R}_{\geq 0}$, such that

$$i) \quad p(z \cdot x) = |z| \cdot p(x) \ , \qquad ii) \quad p(x+y) \leq p(x) + p(y) \ .$$

For p to be a norm, positive definiteness is missing.

Definition 1.1.1.
Let P denote a set of semi normes on X and $F \subset P$ a finite subset. For F and $\varepsilon > 0$ we define:

$$U_{F,\varepsilon}(x) \coloneqq \{y \in X \mid p(x-y) < \varepsilon \quad \forall\, p \in F\} \ .$$

Furthermore we make the convention $U_{F,\varepsilon}(0) = U_{F,\varepsilon}$ and set

$$\mathcal{U} = \{U_{F,\varepsilon} \mid F \subset P \text{ finite}, \, \varepsilon > 0\} \ .$$

The set $\mathcal{U}$ can be regarded as a substitute of the collection of open balls. For that reason we call it $\mathcal{U}$-system, as it will turn out to be a basis of generalized open balls of a topology. To list its properties, we use the following notation:

$$A + B \coloneqq \{a + b \mid a \in A, \ b \in B\} \qquad \text{and} \qquad \Lambda A = \{\lambda a \mid a \in A, \ \lambda \in \Lambda\} \ .$$

Furthermore we need some terminology.

Definition 1.1.2.
Let $A \subset X$ be a subset (not necessarily a sub space), such that $\Lambda A \subset A$ for $\Lambda = \{\lambda \in \mathbb{R} \mid |\lambda| \leq 1\}$. The **Minkowski functional** is defined by

$$p_A\colon X \longrightarrow [0, \infty] \ , \qquad p_A(x) = \inf\{\lambda > 0 \mid \tfrac{x}{\lambda} \in A\} \ .$$

A is called **absorbing**, if $p_A(x) < \infty$ for all $x \in X$.

It can be shown that for convex A, the Minkowski functional is sublinear, i.e. $p_A(x+y) \leq p_A(x) + p_A(y)$ (see [Wer11, Lemma III.2.2]).

Definition 1.1.3.
A subset $A \subset X$ is called **circled** if $\lambda A \subset A$ for all $|\lambda| \leq 1$. It is called **absolute convex** if it is convex and circled.

Now we can list the properties of $\mathcal{U}$:

1. $0 \in U$ for all $U \in \mathcal{U}$.
2. $U_{F_1 \cup F_2, \min(\varepsilon_1, \varepsilon_2)} \subset U_{F_1, \varepsilon_1} \cap U_{F_2, \varepsilon_2}$. Thus for any $U_1, U_2 \in \mathcal{U}$ there is a $U \in \mathcal{U}$, such that $U \subset U_1 \cap U_2$.
3. $U_{F, \frac{\varepsilon}{2}} + U_{F, \frac{\varepsilon}{2}} \subset U_{F,\varepsilon}$. Thus for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$, such that $V + V \subset U$.
4. For $\lambda > \frac{1}{\varepsilon} \max_{p \in F} p(x)$ it holds that $x \in \lambda U_{F,\varepsilon}$. Hence every $U \in \mathcal{U}$ is absorbing.
5. It holds that $\lambda U_{F, \frac{\varepsilon}{\lambda}} = U_{F,\varepsilon}$, such that for every $U \in \mathcal{U}$ and $\lambda > 0$ there is a $V \in \mathcal{U}$, such that $\lambda V \subset U$.
6. Every $U \in \mathcal{U}$ is circled.

These properties are enough to define a topology:

Definition 1.1.4.
Let $\mathcal{U}$ be an $\mathcal{U}$-system then

$$\tau \coloneqq \{O \subset X \mid \forall\, x \in O\ \exists\, U \in \mathcal{U} \colon \{x\} + U \subset O\}$$

is called **locally convex topology**.

To give an intuitive explanation. We take a set O and an element of this set, which will act as translation. Then we need to find a generalized open Ball $U \in \mathcal{U}$, which upon translation still is in the set O. In the case of $P = \{\|\cdot\|\}$ for a normed space, the generalized open balls $\mathcal{U}$ will be just the open balls. Then the locally convex topology translates to, that an open set is a set, such that for every point, we can fit in an open Ball around that point, still being contained in the set.

Corollary 1.1.5.
It holds that

$$\{x\} + U_{F,\varepsilon} = U_{F,\varepsilon}(x)\ .$$

Proof 1.1.6.
By definition $y \in \{x\} + U_{F,\varepsilon}$ means there is a $z \in U_{F,\varepsilon}$, such that $y = z + x$. Then

$$p(x - y) = p(x - z - x) = p(z) < \varepsilon \qquad \forall\, p \in F\ .$$

□

Corollary 1.1.7.
For any $U \in \mathcal{U}$ it holds that $U \in \tau$.

Proof 1.1.8.
Let $x \in U = U_{F,\varepsilon}$ and $\delta = \max_{p \in F} p(x) < \varepsilon$. Thus $\varepsilon' = \frac{\varepsilon - \delta}{2} > 0$ with $\varepsilon' + \delta < \varepsilon$, since $2\varepsilon' + \delta = \varepsilon$. Then $\{x\} + U_{F,\varepsilon'} \subset U_{F,\varepsilon}$, because

$$p(x+y) \leq p(x) + p(y) \leq \delta + \varepsilon < \varepsilon \qquad \forall\, y \in U_{F,\varepsilon'} \ .$$

$\square$

So far, topology is just a name for the set τ, but:

Lemma 1.1.9.
τ is a proper topology on X.

Proof 1.1.10.

i) $\emptyset \in \tau$ and $X \in \tau$ are immediate.

ii) Let $O_1, O_2 \in \tau$ and $x \in O_1 \cap O_2$. Then there are $U_i \in \mathcal{U}$ with $\{x\} + U_i \subset O_i$. By property (2) of $\mathcal{U}$, there is an $U \subset U_1 \cap U_2$ and $\{x\} + U \subset O_1 \cap O_2$. Thus $O_1 \cap O_2 \in \tau$.

iii) Let $O_i \in \tau$ for $i \in I$ and $x \in \bigcup_i O_i$. Then there is at least one $j \in I$ with $x \in O_j$. But this means there is an $U_j \in \mathcal{U}$ with $\{x\} + U_j \subset O_j \subset \bigcup_i O_i$. Thus $\bigcup_i O_i \in \tau$.

$\square$

Definition 1.1.11.
A vector space X with topology τ is called **topological vector space** if the addition and scalar multiplication are continuous w.r.t. τ.

As this definition suggests, we want to show that the locally convex topology promotes X to a topological vector space.

Lemma 1.1.12.
In the locally convex topology τ the maps:

i) $X \times X \longrightarrow X \ , \qquad (x,y) \longmapsto x + y$

ii) $\mathbb{K} \times X \longrightarrow X \ , \qquad (z,x) \longmapsto zx$

are continuous for the product topologies of $X \times X$ and $\mathbb{K} \times X$.

Proof 1.1.13.

i) For $\mathcal{O} \in \tau$ it has to be shown that

$$\mathcal{O}_+ := \{(x, y) \in X \times X \mid x + y \in \mathcal{O}\}$$

is open in the product topology.

Let $(x, y) \in \mathcal{O}_+$. Choose $U \in \mathcal{U}$, such that $U(x + y) \subset \mathcal{O}$ and $V \in \mathcal{U}$, such that $V + V \subset U$ (existence ensured by 3)). Then $(\{x\} + V) \times (\{y\} + V) = V(x) \times V(y) \subset \mathcal{O}_+$. Hence $\mathcal{O}_+$ is open.

ii) For $\mathcal{O} \in \tau$ it has to be shown that

$$\mathcal{O}_\times := \{(\lambda, x) \in \mathbb{K} \times X \mid \lambda x \in \mathcal{O}\}$$

is open in the product topology.

Let $(\lambda, x) \in \mathcal{O}_\times$. Choose $U \in \mathcal{U}$, such that $U(\lambda x) \in \mathcal{O}$ and $V \in \mathcal{U}$ as before. Let $\varepsilon < 0$, such that $\varepsilon x \in V$. Since the sets in $\mathcal{U}$ are circled, it holds that

$$(\mu - \lambda)x \in V \qquad \forall\, \mu\colon |\lambda - \mu| < \varepsilon\ .$$

Choose $W \in \mathcal{U}$ by properties 5) and 6), such that

$$\mu W \subset V \qquad \forall\, \mu\colon |\mu| \leq |\lambda| + \varepsilon\ .$$

For $|\lambda - \mu| < \varepsilon$ and $w \in W$ it follows that

$$\mu \cdot (x + w) - \lambda x = (\mu - \lambda)x + \mu w \in V + V \subset U\ .$$

This shows that $B_\varepsilon(\lambda) \cdot W(x) \subset U(\lambda x)$ and thus $B_\varepsilon(\lambda) \times W(x) \subset \mathcal{O}_\times$. Hence $\mathcal{O}_\times$ is open. □

Owing to this lemma, we can define:

Definition 1.1.14.
A vector space X together with a locally convex topology τ is called **locally convex space** (X, τ).

For a locally convex space it can be shown that:

Lemma 1.1.15.
Let (x, τ) be a locally convex space with topology generated by P then the following claims are equivalent:

i) (X, τ) is a Hausdorff space.

ii) For $x \neq 0$ there is a $p \in P$ with $p(x) \neq 0$.

iii) There is a $\mathcal{U}$-system such that $\bigcap\limits_{U\in\mathcal{U}} U = \{0\}$.

1.2 Continuous functionals

As with Banach spaces, the discussion of locally convex spaces is proceeded by continuous functionals on these spaces. Before we do so, we repeat a result of point set topology connecting different definitions of continuity.

Lemma 1.2.1.
A function $f\colon (X,\tau) \to (Y,\eta)$ is called continuous in $x \in X$, if for every neighborhood V_Y of $f(x)$ there is a neighborhood V_X of x, such that $f(V_X) \subset V_Y$.
A function is continuous if and only if it is continuous for every $x \in X$.

Proof 1.2.2.
The usual definition of continuous functions is, that $f^{-1}(U_Y) \in \tau$ for every $U_Y \in \eta$. The obvious direction is, that a continuous function is continuous for every $x \in X$. This is because neighborhoods V_X can be restricted to open sets U_X by definition. Hence, for every open set $\eta \ni U_Y \ni f(x)$, there is an open set, namely $\tau \ni f^{-1}(U_Y) \ni x$ with $f(f^{-1}(U_Y)) \subset U_Y$.

On the other hand let $U_Y \ni f(x)$ be an open set, i.e. a special neighborhood. Then there is a neighborhood V_X of x, such that $f(V_X) \subset U_Y$. Choose U_X to be an associated open set around x with $U_X \subset V_X$. In fact, to avoid the axiom of choice, we could define $\mathcal{N}$ as the set of neighborhoods V_X of x with $f(V_X) \subset U_Y$ and $\mathcal{O}$ as the set of open sets $U_X \subset V_X$ with $x \in U_X$. Then

$$O_x \coloneqq \bigcup_{V_X\in\mathcal{N}} \bigcup_{U_X\in\mathcal{O}} U_X$$

is an open set with $x \in O_x$ and $f(O_x) \subset U_Y$, that does not require the axiom of choice. The property $f(O_x) \subset U_Y$ can be rewritten as $O_x \subset f^{-1}(U_Y)$, such that we find

$$f^{-1}(U_Y) \subset \bigcup_{x\in f^{-1}(U_Y)} O_x \subset f^{-1}(U_Y)$$

$$\Rightarrow \qquad f^{-1}(U_Y) = \bigcup_{x\in f^{-1}(U_Y)} O_x \ \in \tau \ .$$

□

The following lemma is the basis for a lot of proves concerning continuous functionals:

Lemma 1.2.3.
Let (X,τ) be a locally convex space with topology generated by P.

a) For a semi norm $q\colon X \to [0,\infty)$ the following claims are equivalent:

i) q is continuous.

ii) q is continuous in 0.

iii) $\{x \in X \mid q(x) < 1\}$ is a neighborhood of 0.

b) All $p \in P$ are continuous.

c) A semi norm q is continuous if and only if there are $M > 0$ and $F \subset P$ finite, such that

$$q(x) \leq M \cdot \max_{p\in F} p(x) \qquad \forall\, x \in X\ .$$

Proof 1.2.4.

a) The direction i) $\Rightarrow$ ii) is trivial and ii) $\Rightarrow$ iii) immediate by continuity. It remains to show that iii) $\Rightarrow$ i).

Let $y \in X$ and $\varepsilon > 0$ and choose $U = \varepsilon \cdot \{x \mid q(x) < 1\} = \{x \mid q(x) < \varepsilon\}$. Then using the reverse triangle equation:

$$|q(y+x) - q(y)| \leq q(y+x-y) = q(x) < \varepsilon$$

$$\Rightarrow \qquad q(\{y\}+U) \subset \{a \in \mathbb{R} \mid |a - q(y)| < \varepsilon\} =: V\ .$$

V is a neighborhood of $q(y)$ and $\{y\}+U$ an open neighborhood of $\{y\}$ with $q(\{x\}+U) \subset V$. Then, by lemma 1.2.1 q is continuous.

b) By definition of the locally convex topology, for $F = \{p\}$ the set $\{x \in X \mid p(x) < 1\}$ is a neighborhood of 0. The rest follows from a).

c) By a), the semi norm q is continuous if and only if $V := \{x \in X \mid q(x) < 1\}$ is a neighborhood of 0. This is by definition equivalent to the existence of $F \subset P$ finite and $\varepsilon > 0$, such that $U_{F,\varepsilon} \subset V$, which is equivalent to

$$q(x) \leq \frac{1}{\varepsilon} \cdot \max_{p\in F} p(x) < 1 \qquad \forall\, x \in X\ .$$

□

To highlight that the locally convex topology τ is generated by a family P of semi norms, we write τ_P in the following.

Corollary 1.2.5.
Let (X, τ_P) be a locally convex space an $Q \supset P$ be a super-family of semi norms that are continuous w.r.t. τ_P, then $\tau_P = \tau_Q$.

Proof 1.2.6.
For $q \in Q$ to be continuous it has to hold that there are $F_q \subset P$ and $M_q > 0$, such that $q(x) \leq M_q \cdot \max_{p \in F_q} p(x)$ for all $x \in X$, by lemma 1.2.3. Then for every finite subset $G \subset Q$ we have $M = \max_{q \in G} M_q > 0$ and $F = \bigcup_{q \in G} F_q \subset P$ finite (since G is finite), such that

$$q(x) \leq M \cdot \max_{p \in F} p(x) \qquad \forall\, x \in X,\ q \in G\ .$$

This is leads to $U_{F,\frac{\varepsilon}{M}} \subset U_{G,\varepsilon}$. This is enough to show that, if $O \in \tau_Q$, then also $O \in \tau_P$, i.e. $\tau_Q \subset \tau_P$.

The opposite inclusion $\tau_P \subset \tau_Q$ follows immediately from $\mathcal{U}_P \subset \mathcal{U}_Q$. □

Theorem 1.2.7.
Let (X, τ_P) and (Y, τ_Q) be locally convex spaces and $T\colon X \to Y$ linear, then the following claims are equivalent:

i) T is continuous.

ii) T is continuous in 0.

iii) If q is a continuous semi norm on Y, then $q \circ T$ is a continuous semi norm on X.

iv) For all $q \in Q$ there are $F \subset P$ finite and $M > 0$ such that

$$q(Tx) \leq M \cdot \max_{p \in F} p(x) \qquad \forall\, x \in X\ .$$

A direct corollary for $Y = \mathbb{K}$ with norm topology is

Corollary 1.2.8.
Let (X, τ_P) be locally convex and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be locally convex by $Q = \{\|\cdot\|\}$. A linear map $\ell\colon X \to \mathbb{K}$ is continuous, if and only if there are finitely many $p_1, \ldots, p_n \in P$ and $M > 0$, such that

$$|\ell(x)| \leq M \cdot \max_{i=1,\ldots,n} p_i(x) \qquad \forall\, x \in X\ .$$

Since locally convex spaces are especially topological vector spaces, the concept of topological dual spaces exists without changes. To give the notation, we repeat the definition.

Definition 1.2.9.
Let (X, τ) be a locally convex space. The **dual space** X' is the space of all continuous linear maps $\ell\colon X \to \mathbb{K}$. The set of continuous linear operator $(X, \tau) \to (Y, \eta)$ is denoted by $L(X, Y)$.

To highlight the dependence of the topology, one can write it as subscript, i.e. $(X_\tau)'$ and $L(X_\tau, Y_\eta)$.

Theorem 1.2.10 (**Hahn-Banach theorem**).
Let X be a locally convex space and $U \subset X$ be a sub vector space with $\ell \in U'$. Then there exists an $L \in X'$ with $L|_U \equiv \ell$.

1.3 Weak topologies

To define weak topologies in general, the concept of a dual pair of vector spaces is needed.

Definition 1.3.1.
Let X and Y be vector spaces and $\langle \cdot, \cdot \rangle \colon X \times Y \to \mathbb{K}$ be a $\mathbb{K}$-bilinar map. The pair $(X, Y, \langle \cdot, \cdot \rangle)$ is called **dual pair**, if

$$\forall\, x \in X \setminus \{0\} \;\exists\, y \in Y \colon \langle x, y \rangle \neq 0 \;,$$

$$\forall\, y \in Y \setminus \{0\} \;\exists\, x \in Y \colon \langle x, y \rangle \neq 0 \;.$$

The terminology of dual arises as follows. The map $x \mapsto \ell_x \equiv \langle x, \cdot \rangle$ is linear by linearity of $\langle \cdot, \cdot \rangle$. Thus it is a map $\ell \colon X \mapsto Y^*$, where Y^* denotes the algebraic dual space of Y. Furthermore, the map ℓ is injective, since for $x \neq x'$, it holds that $x - x' \neq 0$ and

$$\ell_x - \ell_{x'} = \langle x, \cdot \rangle - \langle x', \cdot \rangle = \langle x - x', \cdot \rangle \neq 0$$

since by definition of dual pairs, there is a $y \in Y$ with $\langle x - x', y \rangle \neq 0$. Thus $\ell \colon X \hookrightarrow Y^*$ is an injection. The same argument also holds for Y and X^* with the map $y \mapsto \langle \cdot, y \rangle$.

We have purposefully avoided speaking about the topology of X and Y, although the first section was about local convex topologies. This was done, to define a special locally convex topology on dual pairs.

Definition 1.3.2.
Let (X, Y) be a dual pair and $P = \{p_y \mid y \in Y\}$ be the family of semi norms, defined by $p_y(x) = |\langle x, y \rangle|$. The locally convex topology τ_p on X, induced by P, is called $\boldsymbol{\sigma(X, Y)}$**-topology**. Similarly one defines the $\sigma(Y, X)$-topology.

Remark 1.3.3.
By lemma 1.1.15, the $\sigma(X, Y)$ topology is always Hausdorff.

In the special case of $\sigma(X, X^*)$ one speaks about the **weak topology** and in the case of $\sigma(X^*, X)$ one speaks about the **weak-$*$-topology**, not to be confused with the weak topology $\sigma(X^*, X^{**})$ of X^*.

Lemma 1.3.4.
Let X be a vector space, and $\ell_i \colon X \to \mathbb{K}$ be linear functionals for $i = 1, \ldots, n$. Define $N = \{x \in X \mid \ell_i(x) = 0 \;\forall\, i\}$, then the following claims are equivalent:

i) $\ell \in span_{\mathbb{K}}(\ell_1, \ldots, \ell_n)$.

ii) There is $M > 0$*, such that*

$$|\ell(x)| \leq M \cdot \max_{i=1,\dots,n} \ell_i(x) \qquad \forall x \in X \ .$$

iii) $\ell(x) = 0$ *for all* $x \in N$*.*

Proof 1.3.5.
The implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) are immediate. It remains to show that iii) $\Rightarrow$ i).

Define $V := \{(\ell_1(x), \dots, \ell_n(x) \mid x \in X\} \subset \mathbb{K}^n$. Then there is a linear map $\Phi\colon V \to K$, defined by $\ell_i(x) \mapsto \ell(x)$. Indeed, property iii) assures that $\Phi(0) = 0$, such that the map is well defined. From linear algebra, we know that there is an extension $\widehat{\Phi}\colon \mathbb{K}^n \to \mathbb{K}$, that has the form $\widehat{\Phi}(\xi) = \sum_{i=1}^n a_i \xi_i$ for $a_i \in \mathbb{K}$. Thus:

$$\ell(x) = \sum_{i=1}^n a_i \ell_i(x)$$

$$\Rightarrow \qquad \ell = \sum_{i=1}^n a_i \ell_i \quad \Leftrightarrow \quad \ell \in \operatorname{span}_{\mathbb{K}}(\ell_1, \dots, \ell_n) \ .$$

This lemma, though seeming to be applicable to finite dimensions only on first sight, reveals a general property of dual spaces. This property is a direct corollary:

Corollary 1.3.6.
A functional on X *is* $\sigma(X,Y)$*-continuous, if and only if it is of the form* $x \mapsto \langle x, y \rangle$*. Hence* $(X_{\sigma(X,Y)})' = Y$*.* □

Proof 1.3.7.
By corollary 1.2.8 a functional ℓ is continuous, if and only if there are $p_1, \dots, p_n \in P$, such that

$$|\ell(x)| \leq M \cdot \max_{i=1,\dots,n} p_i(x) \qquad \forall\ x \in X \ .$$

However, observing that the semi norms are continuous linear functionals, lemma 1.3.4 can be applied, such that $\ell = \sum_{i=1}^n a_i p_i$ for $a_i \in \mathbb{K}$. Since the $\sigma(X,Y)$-topology is induced by $P = \{p_y \mid y \in Y\}$, the claim follows. □

Theorem 1.3.8.
The weak topology $\sigma(X,Y)$ *is initial w.r.t.* Y*, i.e. if* T *is a topological space the* $f\colon T \to X_{\sigma(X,Y)}$ *is continuous, if and only if all compositions*

$$y \circ f\colon T \xrightarrow{f} X \xrightarrow{y} \mathbb{C} \qquad \text{with} \quad y \in Y$$

are continuous. Furthermore, the weak topology $\sigma(X,Y)$ *is the coarsest topology of* X*, such that* $y \in Y$ *are continuous.*

Proof 1.3.9.
By corollary 1.3.6 it follows that for continuous f the composition $y \circ f$ is continuous, since y is a continuous functional. For the opposite direction, assume $f \circ y$ to be continuous for all $y \in Y$, then it has to be shown that f is continuous. By lemma 1.2.1 this can be done pointwise. Let $t \in T$ and $U \in \mathcal{U}$, then we need to show that there is a neighborhood W of t, such that

$$f(W) \subset f(t) + U \ .$$

Œ we may choose

$$U = \{x \in X \mid |\langle x, y_i \rangle| \leq \varepsilon \ , \ i = 1, \ldots, n\} \ .$$

By assumption $y_i \circ f$ is continuous. Because of corollary 1.3.6 this means, that the map $t \mapsto \langle f(t), y_i \rangle$ is continuous. In terms of continuity in t, this shows, that there are neighborhoods W_i of t, such that

$$|\langle f(t) - f(s), y_i \rangle| = |\langle f(t), y_i \rangle - \langle f(s), y_i \rangle| \leq \varepsilon \quad \forall \ s \in W_i \ .$$

Then, choosing $W = \bigcap_{i=1}^{n} W_i$ it follows that $f(W) \subset f(t) + U$.

The last property follows from choosing $T = X_\tau$, where τ is a topology in which all $y \in Y$ are continuous. Then, considering the identity $\mathrm{Id}\colon X_\tau \to X_{\sigma(X,Y)}$ it follows that Id is continuous, if and only if $\mathrm{Id} \circ y = y$ is continuous. But this is the assumption of τ. Hence, Id is continuous, such that every open set O_X w.r.t. $\sigma(X, Y)$ is open in τ, since $\mathrm{Id}^{-1}(O_X) = O_X$ is open. □

Theorem 1.3.10 (**Banach-Alaoglu theorem**).
Let X be a Banach space. Then the closed unit ball $B_1(0) \subset X'$ is compact in the weak-$*$-topology.

Since any closed subset of a compact set is itself compact and also a finite union of compact sets is compact, there do exist different versions of the Banach-Alaoglu theorem. Furthermore, the concept of **relatively compact** sets, i.e. bounded sets, whose closure is compact, allows for the following corollary:

Corollary 1.3.11.
Let X be a Banach space and $U \subset B_1(0) \subset X'$. Then U is relatively compact in the weak-$*$-topology.

1.4 Banach algebras

We recall, that an associative Algebra is a vector space together with a bilinear operation $\circ$ that is associative, but need not have an inverse or unit element. Most of the time one simply writes ab instead of $a \circ b$

Definition 1.4.1.
Let A be an associative algebra over $\mathbb{C}$ and $\|\cdot\|$ be a norm from the vector space structure. $(A, \|\cdot\|)$ is called **Banach algebra**, if it is complete w.r.t. the norm and

$$\|ab\| \leq \|a\| \cdot \|b\| \ , \qquad \forall\ a, b \in A\ .$$

A an element $\mathbb{1}$ is called **unit element**, if

$$\forall\ a \in A\colon a\mathbb{1} = \mathbb{1}a = a \qquad \text{and} \qquad \|\mathbb{1}\| = 1\ .$$

An algebra with unit element is called **unital**.

Remark 1.4.2.
The condition $\|ab\| \leq \|a\| \cdot \|b\|$ makes the product continuous in the norm topology.

1.4.1 Spectrum of a Banach algebra

Most of this subsection closely follows [All17, p. 6-9].

Definition 1.4.3.
Let A be a unital algebra. The **spectrum** of $a \in A$ is defined by

$$\sigma_A(a) \coloneqq \{z \in \mathbb{C} \mid z \cdot \mathbb{1} - a \text{ can't be inverted in } A\} \subset \mathbb{C}\ .$$

The **spectral radius** is defined as

$$\rho_A \coloneqq \sup |\sigma_A(a)|\ .$$

From this definition follows a corollary about the spectrum of products:

Corollary 1.4.4.
Let A be a unital Banach algebra and $a, b \in A$, then

$$\sigma_A(ab) \setminus \{0\} = \sigma_A(ba) \setminus \{0\}\ .$$

Proof 1.4.5.
Choose $\lambda \in \mathbb{C} \setminus \sigma_A(ab)$ such that $\lambda \neq 0$. Then $\lambda\mathbb{1} - a$ is invertible. Let $c = (\lambda\mathbb{1} - a)^{-1}$. Since

$$c(\lambda\mathbb{1} - ab) = \lambda c - cab = \lambda c - abc = (\lambda\mathbb{1} - a) = e$$
$$\Rightarrow \qquad cab = abc\ ,$$

it holds that:

$$\begin{aligned}(\mathbb{1} + bca)(\lambda\mathbb{1} - ba) &= \lambda\mathbb{1} - \lambda ba + \lambda bca - bcaba \\ &= \lambda\mathbb{1} - \lambda ba + \lambda bca - babca\end{aligned}$$

$$= (\lambda \mathbb{1} - ba)(\mathbb{1} + bca) \ .$$

Furthermore

$$\begin{aligned}
(\mathbb{1} + bca)(\lambda \mathbb{1} - ba) &= \lambda \mathbb{1} - \lambda ba + \lambda bca - bcaba \\
&= \lambda \mathbb{1} - \lambda ba + \lambda bca - babca \\
&= \lambda \mathbb{1} - \lambda ba + bc(\lambda \mathbb{1} - ab)a \\
&= \lambda \mathbb{1} - \lambda ba + b(\lambda \mathbb{1} - ab)^{-1}(\lambda \mathbb{1} - ab)a \\
&= \lambda \mathbb{1} - ba + ba = \lambda \mathbb{1} \ .
\end{aligned}$$

Thus $(\lambda \mathbb{1} - ba)$ is invertible, hence $\mathbb{C} \setminus \sigma_A(ba) \ni \lambda \in \mathbb{C} \setminus \sigma_A(ba)$. This shows that $\sigma_A(ab) \setminus \{0\} = \sigma_A(ba) \setminus \{0\}$. □

For the prove of the next lemma we need Liouville's theorem from complex analysis, stating that:
A function $f\colon \mathbb{C} \to \mathbb{C}$, holomorphic on $\mathbb{C}$ (also called entire function), such that there is an $M \in \mathbb{R}$ so that $\|f(z)\| \leq M$ for all $z \in \mathbb{C}$, then f is constant.

Remark 1.4.6.
Also, we will consider complex analysis on Banach algebras. Most properties from complex analysis carry over without or with minor changes. As example we consider the series

$$\sum_{n=0}^{\infty} a^n z^{-n-1} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \ ,$$

under the assumption that $z \cdot \mathbb{1} - a$ is invertible. Using the continuity of the norm and the Banach algebra property, we find:

$$\left\| \sum_{n=0}^{\infty} a^n z^{-n-1} \right\| \leq \sum_{n=0}^{\infty} \|a\|^n |z|^{-n-1} = \|a\|^{-1} \sum_{n=0}^{\infty} \left(\frac{|z|}{\|a\|}\right)^{-n-1} \ .$$

Hence, by the geometric series, the series $\sum_{n=0}^{\infty} a^n z^{-n-1}$ converges absolutely, for all $|z| > \|a\|$. Although a well know result from analysis, we take an extra step for the geometric series formula, to verify the applicability in the context of Banach algebras. Let $S_N = z \cdot \sum_{n=0}^{N} a^n z^{-n-1}$, then:

$$S_N - \frac{a}{z} S_N = S_N \left(\mathbb{1} - \frac{a}{z}\right) = 1 - \left(\frac{a}{z}\right)^{N+1}$$

$$\Rightarrow \quad S_N = z \cdot 1 - \left(\frac{a}{z}\right)^{N+1} (z \cdot \mathbb{1} - a)^{-1} \overset{N \to \infty}{\longrightarrow} z \cdot (z \cdot \mathbb{1} - a)^{-1}$$

$$\Rightarrow \quad f(z) := \sum_{n=0}^{\infty} a^n z^{-n-1} = \frac{1}{z} S_\infty = (z \cdot \mathbb{1} - a)^{-1}$$

is a holomorphic function for $|z| > \|a\|$, as it has an absolute convergent power series .

Lemma 1.4.7.
Let A be a unital Banach algebra (or a not necessarily unital C^*-algebra). Then $\sigma_A(a) \neq \emptyset$ and $\sigma_A(a)$ is compact for all $a \in A$. Also, the function

$$R_a \colon \mathbb{C} \setminus \sigma_A(a) \longmapsto A \ , \qquad z \longmapsto (z \cdot \mathbb{1} - a)^{-1} \ ,$$

*called **resolvent** is holomorphic. Furthermore it holds that $\rho_A(a) \leq \|a\|$.*

Proof 1.4.8.
As we have seen in remark 1.4.6, the function $R_a(z) = (z \cdot \mathbb{1} - a)^{-1}$ is holomorphic for $|z| > \|a\|$. But then, $\sigma_A(a) \subset B_{\|a\|}(0)$ and it follows that $\rho_A(a) \leq \|a\|$.

Assume now that $\sigma_A(a) = \emptyset$, then R_a is an entire function. It holds that

$$\|R_a(z)\| \leq \sum_{n=0}^{\infty} \|a\|^n |z|^{-n-1} = \frac{1}{|z| - \|a\|} \overset{|z|\to\infty}{\longrightarrow} 0$$

$$\Rightarrow \qquad R_a(z) \longrightarrow 0 \ , \quad |z| \to \infty \ .$$

But since R_a is an entire function, Liouville's theorem states that R_a is identical 0, which is obviously a contradiction. Hence $\sigma_A(a) \neq \emptyset$.

Since compactness equals closed and bounded (Heine Borel theorem in $\mathbb{C}$) we only need to show that $\sigma_A(a)$ is closed. Choose $z_0 \not\in \sigma_A(a)$, such that $z_0 \cdot \mathbb{1} - a$ is invertible. For $z \in \mathbb{C}$ with

$$|z - z_0| < \frac{1}{\|(z_0 \cdot \mathbb{1} - a)^{-1}\|}$$

the series

$$\sum_{n=0}^{\infty} (z_0 \cdot \mathbb{1} - a)^{-n-1} (z - z_0)^n = (z_0 \cdot \mathbb{1} - a) \sum_{n=0}^{\infty} ((z - z_0) \cdot (z_0 \cdot \mathbb{1} - a))^n$$

converges absolutely against $(z - a)^{-1}$. This can be verified with the same methods as in remark 1.4.6. But this means, that for any $z_0 \in \mathbb{C} \setminus \sigma_A(a)$ the function R_a is holomorphic in a neighborhood around z_0. Put differently $\mathbb{C}/\sigma_A(a)$ is open and by definition $\sigma_A(a)$ the closed. □

Following [Con97, VII, 5.4 Theorem], we can show, how the spectra of Banach subalgebras relate to the original Banach algebra.

Theorem 1.4.9.
Let A, B be unital Banach algebras with $B \subset A$, such that $\mathbb{1} \in A$ and $\mathbb{1} \in B$, then

$$\sigma_A(a) \subset \sigma_B(a) \qquad \text{and} \qquad \partial\sigma_B(a) \subset \partial\sigma_A(a) \ .$$

Proof 1.4.10.
The first inclusion follows by definition. For the second inclusion, let $\lambda \in \partial\sigma_B(a)$. By the previous inclusion, it also holds $\text{int}\sigma_A(a) \subset \text{int}\sigma_B(a)$, where $\text{int}X$ denotes the interior of a set X. Thus, if $\lambda \in \sigma_A(a)$ it follows that $\lambda \in \partial\sigma_A(a)$.

Assume $\lambda \not\in \sigma_A(a)$. This means $(\lambda\mathbb{1} - a)^{-1} \in B$. Since $\lambda \in \partial\sigma_B(a)$, there is a sequence (λ_n) with $\lambda_n \to \lambda$ and $\lambda_n \in \mathbb{C} \setminus \sigma_B(a)$. Thus $(\lambda_n\mathbb{1} - a)^{-1}$ exists in B and hence also in A. From $\lambda_n \to \lambda$ it follows that $\lambda_n\mathbb{1} - a \to \lambda\mathbb{1} - a$, but then $(\lambda_n\mathbb{1} - a)^{-1} \to (\lambda\mathbb{1} - a)^{-1}$. So $(\lambda\mathbb{1} - a)^{-1} \in B$ by completeness of B. However, this contradicts $\lambda \in \sigma_B(a)$. □

Lemma 1.4.11.
Let $p \in \mathbb{C}[z]$ be a polynomial of the variable z with coefficients in $\mathbb{C}$ and $a \in A$, then it holds that

$$\sigma_A(p(a)) = p(\sigma_A(a)) \ .$$

Proof 1.4.12.
Let $\omega \in C$ and consider $\omega - p(z)$. Since this again is polynomial, it has roots z_j, such that

$$\omega - p(z) = \prod_j (z - z_j) \ .$$

Put differently, the z_j are solutions of $p(z) = \omega$, such that
$\{z_1, \ldots, z_n\} = p^{-1}(\omega)$. Passing to a polynomial with variables in A, we obtain

$$\omega \cdot e - p(a) = \prod_j (a - z_j \cdot \mathbb{1}) \ .$$

Then, $\omega \cdot \mathbb{1} - p(a)$ is not invertible, i.e. $\omega\sigma_A(p(a))$, if there is any z_j, for which $a - z_j \cdot \mathbb{1}$ is not invertible. The existence of such an $z_j \in p^{-1}(\omega)$ means

$$z_j \in \sigma_A(a) \quad \text{and} \quad z_j \in p^{-1}(\omega) \qquad \Leftrightarrow \qquad \sigma_A(a) \cap p^{-1}(\omega) \neq \emptyset \ .$$

This means, that $\omega = p(z_j) \in p(\sigma_A(a))$. □

Corollary 1.4.13.
It holds that

$$\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \ .$$

Proof 1.4.14.
From lemma 1.4.7 we know that $R_a(z)$ is holomorphic for $|z| > \rho_A(a)$. Furthermore, the power series of R_a converges absolutely for $|z| > \rho_A(a)$, as we have seen in remark 1.4.6. As a consequence of the absolute convergence of the series, it holds that

$$\lim_{n\to\infty} \|a^n\| r^{-n-1} = 0 \qquad \forall\, r > \rho_A(a) \ .$$

Using $\frac{1}{z}$ instead of z (up to a factor of z), the power series $\sum_{n=0}^{\infty} \|a^n\| z^n$ has the radius of convergence $R \geq \frac{1}{\rho_A(a)}$. By the Cauchy–Hadamard theorem this means:

$$\rho_A(a) \geq \frac{1}{R} = \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \ .$$

Also from lemma 1.4.7 it follows that $\sigma_A(a)$ is non-empty and compact. Thus there is a $z \in \sigma_A(a)$ such that $|z| = \rho_A(a)$. By lemma 1.4.11 it holds that $z^n \in \sigma_A(a^n)$. Using lemma 1.4.7 again, i.e. $\rho_A(a) \leq \|a\|$, we find $|z^n| = \|a^n\|$ and thus

$$\rho_A(a) = |z^n|^{\frac{1}{n}} \leq \|a^n\|^{\frac{1}{n}} \ .$$

But since n has not been specified,

$$\rho_A(a) \leq \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}} \ .$$

Together with $\rho_A(a) \geq \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}}$, the claim follows. ☐

There is a last result concerning spectra of Banach algebras from [Wer11] that we need, in order to investigate maximal spectra and Gelfand spaces.

Theorem 1.4.15 (**Gelfand–Mazur theorem**).
A unital Banach algebra, where every non-zero element is invertible, is one dimensional and commutative, i.e. $A = \mathbb{C} \cdot \mathbb{1}$.

Proof 1.4.16 ([Wer11, p. 479]).
Let $a \in A$ and choose $\lambda \in \sigma_A(a)$, which is possible since $\sigma_A(a) \neq \emptyset$ by lemma 1.4.7. By definition, $\lambda \cdot \mathbb{1} - a$ is not invertible. By assumption, it has to hold then, that $\lambda \cdot \mathbb{1} - a = 0$, hence $a = \lambda \cdot \mathbb{1}$. ☐

1.4.2 Maximal spectrum

This subsection follows [Wer11, chapter IX.2]

An algebra homomorphism is a linear map between algebras, that are also a homomorphism w.r.t. the multiplication. A special kind of algebra homomorphisms are functionals $\varphi \colon A \to \mathbb{C}$, since $\mathbb{C}$ carries a natural algebra structure as field.

Lemma 1.4.17.
Let A be a Banach algebra, then every algebra homomorphism $\varphi \colon A \to \mathbb{C}$ is continuous and $\|\varphi\| \leq 1$. If A is unital, it holds that $\|\varphi\| = 1$ or $\|\varphi\| = 0$.

Proof 1.4.18.
Assume $1 < \|\varphi\| \leq \infty$ and assume $\varphi \neq 0$, since $\|0\| = 0$ is trivial. This means, that

there for $x \in A$ with $\varphi(x) = 1$, it holds that $\|x\| < 1$. This can always be achieved by taking $x = \frac{1}{\varphi(x_0)} \cdot x_0$. Considering $y = x + yx$, e.g. $y = \sum_{n=0}^{\infty} x^n$ it follows that

$$\varphi(y) = \varphi(x) + \varphi(x)\varphi(y) = 1 + \varphi(y) \ ,$$

which is a contradiction. Thus $\|\varphi\| \leq 1$. By theorem 1.2.7 (iv) this shows that φ is continuous, using the norm topology τ_P for $P = \{\|\cdot\|\}$.

In case of unital Banach algebras it follows that (for $\varphi \neq 0$):

$$1 = \varphi(x) = \varphi(\mathbb{1}x) = \varphi(\mathbb{1})\varphi(x) = \varphi(\mathbb{1}) \ .$$

Thus from $|\varphi(\mathbb{1})| = 1$ and $\|\mathbb{1}\| = 1$ it follows that $\|\varphi\| \geq 1$ by the supremum definition. Together with $\|\varphi\| \leq 1$ from the general case, it follows that $\|\varphi\| = 1$. □

For the next lemma, we make the observation, that an associative algebra has a ring structure with additional scalar multiplication from the vector space (alternative definition).

Definition 1.4.19.
A subspace of a an algebra A is called (two sided) **ideal**, if

i) $(I, +)$ is a proper sub group ,

ii) $x \circ a, \ a \circ x \in I \quad \forall\, x \in A, a \in I$.

The ideal is called **proper** if $I \neq A$. An ideal I is called **maximal**, if it is proper and fro every ideal J with $I \subsetneq J$ it follows that $J = A$.

For properties of ideals in Banach algebras we need the Neumann series.

Theorem 1.4.20 (**Neumann series**).
Let A be a unital Banach algebra. If $\|a\| < 1$, then $\mathbb{1} - a$ is invertible and

$$(\mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n \ ,$$

where $a^0 = \mathbb{1}$.

Proof 1.4.21 (From [Wer11, Satz II.1.11]).
Define $S_m = \sum_{n=0}^{m} a^n$. Then it holds that (see remark 1.4.7 for a slightly more general version)

$$(\mathbb{1} - a)S_m = S_m(\mathbb{1} - a) = \mathbb{1} - a^{m+1} \ .$$

Since $\|a\| < 1$, the series converges absolutely. By continuity of the product in Banach algebras, we find

$$\mathbb{1} = \lim_{m\to\infty} (\mathbb{1} - a^{m+1}) = \lim_{m\to\infty} (\mathbb{1} - a)S_m = (\mathbb{1} - a) \lim_{m\to\infty} S_m$$

$$= (\mathbb{1} - a) \sum_{n=0}^{\infty} a^n \ ,$$

and also

$$\mathbb{1} = \left(\sum_{n=0}^{\infty} a^n \right) (\mathbb{1} - a) \qquad \Rightarrow \qquad (\mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n \ .$$

□

Corollary 1.4.22.
Let $A^\times$ denote the set of invertible elements of A and let $x \in A^\times$. Then, for all $h \in A$ with $\|h\| < \frac{1}{\|x^1\|}$ it holds that $x + h \in A^\times$. Thus $A^\times$ is open w.r.t. the norm topology.

Proof 1.4.23.
Rewriting $x + h$ yields

$$x + h = x(e + x^{-1}h) \ .$$

By theorem 1.4.20 $e + x^{-1}h$ is inverible, because

$$\|x^{-1}h\| \leq \|x^{-1}\| \cdot \|h\| < \|x^{-1}\| \cdot \frac{1}{\|x^1\|} = 1 \ .$$

Hence, as a product of two invertible elements $x + h$ is invertible. □

Lemma 1.4.24.
Let A be a Banach algebra, then it holds that:

i) The closure of an ideal is again an ideal.

ii) The quotient A/I for a closed ideal I is again a Banach algebra with the quotient norm[1], where $[a][b] = [ab]$. If A is commutative, so is A/I.

For the following let A be unital, too:

iii) A proper ideal is not dense in A.

iv) Maximal ideals are closed.

v) If I is a proper ideal, then A/I is unital.

vi) If I is a maximal ideal and A commutative, then A/I is one dimensional.

[1] The closedness of I is necessary for the quotient norm to be a proper norm:

$$\|a\|_{A/I} = \|a + I\| := \inf\{\|a - x\| \mid x \in I\} \ .$$

Proof 1.4.25.

i) Let $a \in \overline{I}$ and $y \in A$. There is a sequence (x_n) with $x_n \in I$ and $x_n \to x$. By continuity of the product in Banach algebras it follows from $x_n y \in I$ and $y x_n \in I$, that $xy \in \overline{I}$ and $yx \in \overline{I}$.

ii) By definition $a \in [b]$, if $a + x = b$ for $x \in I$. The rest are direct calculations.

iii) Assume I to be a proper ideal with $\overline{I} = A$. Then there is a sequence $a_n \in I$, such that $a_n \to \mathbb{1}$. Being a proper ideal means that $\mathbb{1} \notin I$. However, there is an $m \in \mathbb{N}$ with $\|e - a_m\| < 1$ by definition of convergence. Rewriting the Neumann series from theorem 1.4.20, choosing $\mathbb{1} - a_m$ with $\|e - a_m\| < 1$ instead of a shows that a_m is invertible. The inverse $a_m^{-1} \in A$ need not be in I, but $a_m a_m^{-1} = \mathbb{1} \in I$ is by definition of ideals a necessity. Yet this is a contradiction. Thus, a proper ideal is not dense in A.

iv) This is a combination of i) and iii) and the definition of maximal ideals, i.e. $I \subsetneq \overline{I}$ yields $\overline{I} = A$.

v) The unit of ${}^{A}/_{I}$ is given by $[\mathbb{1}] \neq [0]$.

vi) From ii), iv) and v) it follows that ${}^{A}/_{I}$ is a unital commutative Banach algebra. Using the Gelfand-Mauzer theorem 1.4.15, it is enough to show that

$$\forall\, [x] \in {}^{A}/_{I}\ \exists\, [a] \in {}^{A}/_{I} \colon [x][y] = [xa] = [\mathbb{1}]\ .$$

This holds true, if one finds $a \in A$ for all $[x] \in {}^{A}/_{I}$, such that $[xa] = [\mathbb{1}]$. For $[x] \in {}^{A}/_{I}$ define

$$J_x := \{xa + b \mid a \in A, b \in I\} \subset A\ .$$

This set is well defined and does not depend on the representative of $[x]$, since $(x + c)a + b = xa + (ca + b)$ with $(ca + b) \in I$. It can be checked that J_x is an ideal (here commutativity is used). Also, $I \subsetneq J_x$ (choose $a = 0$ and $a = \mathbb{1}$ with $b = 0$), but by assumption I is maximal, hence $J_x = A$. This means, there are $a \in A$ and $b \in I$, such that $xa + b = \mathbb{1}$, i.e. $[xa] = [\mathbb{1}]$.

□

Definition 1.4.26.
Let A be a Banach algebra, then

$$\Gamma_A := \{\varphi \colon A \to \mathbb{C} \mid \varphi \neq 0 \text{ and } \varphi \text{ is an algebra homomorphism}\}$$

is called the **maximal spectrum** of A.
The map $\Gamma \colon a \mapsto \Gamma(a)$ defined by $\Gamma(a)(\varphi) = \varphi(a)$ for $a \in A$ and $\varphi \in \Gamma_A$ is called **Gelfand transformation**.

By lemma 1.4.17, elements of Γ_A are continuous, such that $\Gamma_A \subset A'$. Hence Γ_A can be equipped with the subspace weak-$*$-topology $\sigma(A', A)$. We call $(\Gamma_A, \sigma(A', A))$ **Gelfand space** and write Γ_A for short.

Theorem 1.4.27.

i) The Gelfand space Γ_A is a relative compact Hausdorff space and $\Gamma(a) \in C(\Gamma_A)$, i.e. $\Gamma\colon A \to C(\Gamma_A)$. If A is unital, then Γ_A is compact and $\Gamma(a) \in C_0(\Gamma_A)$.

ii) Let A be commutative and unital, then $I \subset A$ is a maximal ideal, if and only if $I = \mathrm{Ker}(\varphi)$ for a $\varphi \in \Gamma_A$.

Remark 1.4.28.
The space $C(X)$ denotes the space of continuous functions on X. For a locally compact Hausdorff space, $C_0(X)$ is the space of all continuous functions $f\colon X \to \mathbb{C}$, such that for all $\varepsilon > 0$ the set $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact (see [Con97, 1.7 Proposition]). In case of compact spaces this is always the case for $f \in C(X)$.
A subset $A \subset X$ of a topological space X is **relatively compact**, if $\overline{A}$ is compact in X. A topological space X is **locally compact** if for every $x \in X$ there are a compact set $K \subset X$ and an open set $U \subset X$, such that $x \subset U \subset K$. A relative compact set is locally compact in the subspace topology.

Proof 1.4.29.

i) By lemma 1.4.17 it holds that $\Gamma_A \subset B_1(0) \subset A'$. Hence by the Banach-Alaoglu theorem 1.3.10, Γ_A is relatively compact in the weak-$*$-topology. Also by lemma 1.1.15, every space with weak-$*$-topology is Hausdorff. Observing, that the map $a \mapsto \Gamma(a)$ is but a restriction of $a \mapsto \langle \cdot, a\rangle$, the continuity of $\Gamma(a)$ follows from corollary 1.3.6. Since the zero map is excluded from Γ_A one can consider the **one-point compactification** $\dot{\Gamma}_A \coloneqq \Gamma_A \cup \{0\}$ with topology

$$\tau = \{U \subset \Gamma_A \mid U \text{ is open}\} \cup \{\dot{\Gamma}_A \setminus K \mid K \subset \Gamma_A \text{ is compact}\}\ .$$

The map $\langle \cdot, a\rangle =: f_a$ is continuous on $A' \supset \dot{\Gamma}_A$ and hence continuous in 0 by lemma 1.2.1. Furthermore, $f_a(0) = \langle 0, a\rangle = 0$. Being continuous in 0 on the locally compact space means $f_a^{-1}(B_\varepsilon(f_a(0)))$ is open in $\dot{\Gamma}_A$. Put differently:

$$\forall\, \varepsilon > 0\ \exists\ K \subset \Gamma_A \text{ compact}\ :$$

$$|f_a(x) - f_a(0)| = |f_a(x)| < \varepsilon \qquad \forall\ x \in \Gamma_A \setminus K\ .$$

This shows that $\{x \in \Gamma_A \mid |f_a(x)| > \varepsilon\}$ is compact. Hence $f_a|_{\Gamma_A} = \Gamma(a) \in C_0(\Gamma_A)$.

Assume now A to be unital. We only need to show that Γ_A is closed in the weak-$*$-topology, following [Con97, proof of Thmeorem VII 8.6.]. Assume $\varphi \in B_1(0) \subset A'$ and let (φ_n) be a sequence in Γ_A such that $\varphi_n \to \varphi$ in the weak-$*$-topology. Then, for $a, b \in A$:

$$\varphi(ab) = \lim_{n\to\infty} \varphi_n(ab) = \lim_{n\to\infty} \varphi_n(a)\varphi_n(b) = \varphi(a)\varphi(b)$$

$$\text{and} \quad \varphi(\mathbb{1}) = \lim_{n\to\infty} \varphi_n(\mathbb{1}) = 1\ .$$

Thus φ is an algebra homomorphism with $\|\varphi\| = 1$ and hence $\varphi \in \Gamma_A$, which shows that Γ_A is closed.

ii) Let I be a maximal ideal of A. Then by lemma 1.4.24, property vi), it holds that ${}^{A}/_{I} \cong \mathbb{C}$. Building the quotient is a homomorphism $\pi_I \colon A \to \mathbb{C}$, and its kernel is I. Thus every maximal ideal is a kernel of $\varphi = \pi_I \in \Gamma_A$.

The opposite direction follows from the fundamental theorem of homomorphisms for rings. I.e. $\mathrm{Ker}(\varphi)$ is an ideal and ${}^{A}/_{\mathrm{Ker}(\varphi)} \cong \mathrm{Im}(\varphi) \subset \mathbb{C}$. Hence $\mathrm{Ker}(\varphi)$ has codimension one, and thus is maximal.

□

1.5 Stone-Weierstrass theorem

In this section we want to state the Stone-Weierstrass theorem as given in [Con97, Chapter V, section 8].

Definition 1.5.1.
Let A be a sub algebra of $C(X)$, then A is said to **separate the points** of X, if for $x, y \in X$ with $x \neq y$ there is a function $f \in A$, such that $f(x) \neq f(y)$.

As usual, we assume f to be complex valued, allowing to define $\overline{f}$ by $\overline{f}(x) = \overline{f(x)}$.

Theorem 1.5.2 (**Stone-Weierstrass theorem**).
Let X be a compact Hausdorff space and A be a closed subalgebra of $C(X)$ that separates the points of X. If for $f \in A$ also $\overline{f} \in A$, then $A = C(X)$.

There exists also a version of the Stone-Weierstrass theorem for C^*-algebras (see [All17, A.2]):

Theorem 1.5.3 (**Stone-Weierstrass theorem** for C^*-algebras).
Let X be a compact Hausdorff space and A be a unital C^*-subalgebra of $C(X)$ that separates the points of X, then $A = C(X)$.

A consequence of this theorem is:

Corollary 1.5.4.
Let X be a locally compact Hausdorff space and $A \subset C_0(X)$ a C^*-sub algebra. If A separates the points of X and also for all $x \in X$ there is an $f \in A$, such that $f(x) \neq 0$, then it holds that $A = C_0(X)$.

2
C^*-Algebras

C^*-algebras are special Banach algebras with an additional structure, the $*$-operator, an abstract generalization of the concept of adjoint operators $A^\dagger$ of Hilbert spaces. This chapter follows [All17, chapter 1] very closely, with some additional statements from [Mur90] and [Con97].

2.1 Definition of C^*-algebras

Definition 2.1.1.
An associative algebra A over $\mathbb{C}$ is called **$*$-algebra**, if there is a map $*\colon A \to A$, with the properties

$$(a + z \cdot b)^* = a^* + \overline{z} \cdot b^* \ , \qquad (ab)^* = b^* a^* \ , \qquad (a^*)^* = a \ ,$$

for all $a, b \in A$, $z \in \mathbb{C}$. Such a map is called an antilinear anti-involution.

The concepts of $*$-algebra and Banach algebra can me mixed with further constrains to obtain further algebras.

Definition 2.1.2.
A Banach algebra A that is also a $*$-algebra is called **Banach $*$-algebra**, if

$$\|a^*\| = \|a\| \ , \qquad \forall\, a \in A \ .$$

A Banach algebra A that is also a $*$-algebra is called a **C^*-algebra** if

$$\|a^* a\| = \|a\|^2 \ , \qquad \forall\, a \in A \ .$$

The property to be a C^* algebra is stronger than the property to be a Banach $*$-algebra.

Corollary 2.1.3.
Let A be a C^*-algebra, then A is also a Banach $*$-algebra.

Proof 2.1.4.
Use that a C^*-algebra is also a Banach algebra together with the defining property to see that

$$\|a\|^2 = \|a^* a\| \leq \|a^*\| \cdot \|a\| \qquad \Rightarrow \qquad \|a\| \leq \|a^*\| \ .$$

Exchanging a with a^* yields the opposite inequality, such that $\|a\| = \|a^*\|$. $\square$

As for general homomorphisms, we define a **$*$-morphism** to be a map $\phi\colon A \to B$, such that

$$\phi(a^*) = \phi(a)^* \ .$$

As the notation suggests, an element $a \in A$ is called **self adjoint**, if $a^* = a$. Additionally we call an element **normal**, if $a\ a^* = a^* a$.

Corollary 2.1.5.
Every $a \in A$ can be written as sum $a = b + ic$ with self adjoint $b, c \in A$.

Proof 2.1.6.

$$a = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*) \ .$$

□

If A is also unital, then an element $u \in A$ is called **unitary**, if

$$u\ u^* = 1 = u^* u \ .$$

Hence unitary elements are always normal, but not vice versa. By the anti-involutory property, the unit element $\mathbb{1}$ is self adjoint, since

$$a \circ \mathbb{1}^* = ((a \circ \mathbb{1}^*)^*)^* = (\mathbb{1} \circ a^*)^* = (a^*)^* = a \ .$$

Furthermore,

$$\|\mathbb{1}\|^2 = \|\mathbb{1} \circ \mathbb{1}^*\| = \|\mathbb{1}\| \neq 0 \qquad \Rightarrow \qquad \|\mathbb{1}\| = 1 \in \mathbb{C} \ .$$

A direct consequence for an unitary element is

$$\|u\|^2 = \|u^* u\| = \|\mathbb{1}\| = 1 \qquad \Rightarrow \qquad \|u\| = 1 \ .$$

Lemma 2.1.7.
Let A be an arbitrary C^*-algebra, then there is a unital C^*-algebra $\widetilde{A}$, such that A is an ideal of $\widetilde{A}$, closed w.r.t. the norm topology of $\widetilde{A}$. If A is not unital, then it is a maximal ideal, with codimension 1.[1]

Proof 2.1.8.
Let $\mathcal{L}(A)$ denote the set the bounded endomorphisms, becoming an algebra by pointwise addition/scalar multiplication and the composition of maps. Define the map

$$\pi\colon A \longrightarrow \mathcal{L}(A) \ , \qquad a \longmapsto \pi(a) \equiv a \circ \ \ .$$

[1] $\mathrm{codim}(A) = \dim(\widetilde{A}) - \dim(A)$.

This map is, by the bilinearity of $\circ$, an algebra homomorphism. Let $a_1 \neq a_2$ we observe that $a_1^* \neq a_2^*$, since

$$(0 \cdot a)^* = 0 \cdot a^* = 0 \qquad \Rightarrow \qquad 0 = 0^* \neq (a_1 - a_2)^* = a_1^* - a_2^* \ .$$

Assume now $\pi(a_1) = \pi(a_2)$, then

$$0 = \pi(a_1)b - \pi(a_2)b = (a_1 - a_2) \circ b \quad \forall\, b \in A \ .$$

Yet, choosing $b = (a_1 - a_2)^* \neq 0$ leads to a contradiction:

$$\|b^*b\| = \|(a_1 - a_2)b\| = \|0\| \neq \|b\|^0 \ .$$

Thus the assumption was wrong and the map π is injective. A property of Banach spaces is, that complete sub vector spaces are closed.[2]Hence $\pi(A) \subset \mathcal{L}(A)$ is closed.

Furthermore:

$$\begin{aligned}\|\pi(a)\| &= \sup \lim_{\|b\|\leq 1} \|\pi(a)b\| = \sup \lim_{\|b\|\leq 1} \|ab\| \\ &\leq \sup \lim_{\|b\|\leq 1} \|a\| \cdot \|b\| \leq \|a\| \ ,\end{aligned}$$

by the fact that a C^*-algebra is a Banach algebra. Using that A is a C^*-algebra, which is Banach $*$-algebra, we find

$$\|a\|^2 = \|a^*\|^2 = \|a \circ a^*\| = \|\pi(a)a^*\| \leq \|\pi(a)\| \cdot \|a^*\| = \|\pi(a)\| \cdot \|a\| \ .$$

Hence $\|\pi(a)\| = \|a\|$, i.e. π is an isometry.

Define $\widetilde{A}$ to be

$$\widetilde{A} \coloneqq \{\pi(a) + z \cdot \mathbb{1} \mid a \in A, z \in \mathbb{C}\} \ .$$

The elements of $\widetilde{A}$ are bounded endomorphisms of A and the set $\widetilde{A}$ is closed under the algebra operations. Hence $\widetilde{A} \subset \mathcal{L}(A)$ is a sub algebra. Furthermore, since

$$\begin{aligned}\pi(b) \circ (\pi(a) + z\mathbb{1}) &= \pi(b \circ a) + z\pi(b) = \pi(b \circ a + z \cdot b) \\ \text{and} \qquad (\pi(a) + z\mathbb{1}) \circ \pi(b) &= \pi(a \circ b + z \cdot b) \ ,\end{aligned}$$

it follows that $\pi(A)$ is an ideal of $\widetilde{A}$. Depending if A is unital or not, it holds that $\widetilde{A}/\pi(A) = 0$ or $\widetilde{A}/\pi(A) = \mathbb{C}$. In the latter case, where A is not unital, we also have $\operatorname{codim}(\pi(A)) = 1$. The Banach algebra $\widetilde{A}$ can be extended to a Banach $*$-algebra by

$$(\pi(a) + z\mathbb{1})^* \coloneqq \pi(a^*) + \overline{z}\mathbb{1} \ .$$

Finally let $\varepsilon > 0$. By the properties of the operator norm, there is a $b \in A$ with $\|b\| \leq 1$, such that:

$$\begin{aligned}\|\pi(a) + z\mathbb{1}\|^2 &\leq \varepsilon + \|(\pi(a) + z\mathbb{1})b\|^2 \\ &= \varepsilon + \|b^*(\pi(a) + z\mathbb{1})^*(\pi(a) + z\mathbb{1})b\| \\ &\leq \varepsilon\|b\| \cdot \|(\pi(a^*) + \overline{z}\mathbb{1})(\pi(a) + z\mathbb{1})b\| \\ &\leq \varepsilon\|(\pi(a^*) + \overline{z}\mathbb{1})(\pi(a) + z\mathbb{1})b\| \\ &\leq \varepsilon\|(\pi(a^*) + \overline{z}\mathbb{1})(\pi(a) + z\mathbb{1})\| \cdot \|b\|\end{aligned}$$

$$\leq \varepsilon \|(\pi(a^*) + \bar{z}\mathbb{1})(\pi(a) + z\mathbb{1})\| \ .$$

Since ε was chosen arbitrarily, the C^*-property holds for $\widetilde{A}$. Since π is an injective algebra homomorphism, it is an injection, such that we can understand $\pi(A)$ as A in $\widetilde{A}$. □

2.2 Spectrum and maximal spectrum of C^*- algebras

The definitions of spectrum and spectral radius require a unit element. However in case of C^*-algebras, because of the inclusion $A \subset \widetilde{A}$, the definition can be extended to arbitrary C^*-algebras.

Definition 2.2.1.
In case of a non-unital C^*-algebra, one defines $\sigma_A(a) \coloneqq \sigma_{\widetilde{A}}(a)$ and consequently $\rho_A(a) \coloneqq \rho_{\widetilde{A}}(a)$. For that reason one drops the index A.

This definition allows to carry over all results for the spectrum and the spectral radius from unital Banach algebras to C^*-algebras.

Corollary 2.2.2.
Let A be a C^*-algebra and $a \in A$ normal, then $\rho(a) = \|a\|$.

Proof 2.2.3.
Fist we consider a self adjoint $a \in A$. Then

$$\begin{aligned} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a^{2^n}\|^{2^{-n}} &= \left(\left\|(a^{2^{n-1}})^*(a^{2^{n-1}})\right\|^{\frac{1}{2}}\right)^{2^{-n+1}} \\ &= \|a^{2^{n-1}}\|^{2^{-n+1}} = \ldots = \|a\| \ , \end{aligned}$$

where self adjointness was used in the third step. Thus by corollary 1.4.13

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \|a\| \ .$$

Then for a normal a we can use that a^*a is self adjoint:

$$\begin{aligned} \rho(a)^2 \leq \|a\|^2 = \|a^*a\| = \rho(a^*a) &= \lim_{n\to\infty} \|(a^*a)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(a^*)^n a^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|(a^*)^n\|^{\frac{1}{n}} \|a^n\|^{\frac{1}{n}} \end{aligned}$$

[2]The idea is, that the limit point of every convergent sequence is in the subspace $U \subset X$ by completeness. Suppose that $x \in X \setminus U$. If there is no $\varepsilon > 0$ such that $B_\varepsilon(x) \subset X \setminus U$, i.e. for all ε there is a point $p_\varepsilon \in U$ with $\|p_\varepsilon - x\| < \varepsilon$ we would have found a construction for a sequence $(p_n) \in U$ with $\lim_{n\to} p_n = x$. But then x would be an element of U as a limit point → contradiction. Thus there is an $\varepsilon > 0$ for every $x \in X \setminus U$ such that $B_\varepsilon(x) \subset X \setminus U$...

$$= \lim_{n\to\infty} \|(a^*)^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \rho(a)^2$$

Hence

$$\rho(a)^2 = \|a\|^2 \qquad \Rightarrow \qquad \rho(a) = \|a\| \ .$$

□

By theorem 1.4.27 we know that Γ_A is compact for a unital commutative C^*-algebra. Dropping the existence of a unit element, Γ_A is relatively compact. In case of a commutative C^*-algebra, it can even be shown, that the Gelfand transformation is an isometric $*$-isomorphism. Yet the following lemmas are needed for the proof:

Lemma 2.2.4.
Let A be a C^*-algebra. If $a \in A$ is self adjoint, then $\sigma(a) \subset \mathbb{R}$. If A is unital and $u \in A$ is unitary, then $\sigma(u) \subset \mathrm{U}(1)$.

Proof 2.2.5.
Starting with the unital case first, let $u \in A$ be unitary and $z \in \sigma(u)$. It holds that for invertible y, such that $yx = xy$, the element $x \in \tilde{A}$ is not invertible, if and only if yx is not invertible. This is equivalent to x invertible $\Leftrightarrow$ yx invertibe). Then,

$$(-zu)^{-1} = -\frac{1}{z}u^* \quad \Rightarrow \quad \frac{1}{z}u^* \circ (ze - u) = \frac{1}{z} - u^* = (ze - u) \circ \frac{1}{z}u^* \ .$$

Thus $z^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$. From corollary 2.2.2 we deduce:

$$|z| \leq \rho(u) = \|u\| = 1 \qquad \Rightarrow \qquad |z| = 1 \ ,$$

that is, $z \in \mathrm{U}(1)$.

Let $a \in A$ be self adjoint, i.e. $a^* = a$. In $\tilde{A}$ it holds that

$$\exp(ia) = \sum_{n=0}^{\infty} \frac{1}{n!}(ia)^n \ \in \ \tilde{A} \ .$$

Furthermore, $\exp(ia)^* = \exp(-ia) = \exp(ia)^{-1}$, such that $\exp(ia)$ is unitary in $\tilde{A}$. Hence $\sigma(\exp(ia)) \subset \mathrm{U}(1)$. Let $z \in \sigma(a)$ and assume $z \not\in \mathbb{R}$, such that $\exp(iz) \not\in \mathrm{U}(1)$, i.e. $\exp(iz) \not\in \sigma(\exp(ia))$. This means that $p_\infty := \exp(iz) \cdot \mathbb{1} - \exp(ia)$ is invertible in $\tilde{A}$. The sequence

$$p_n = \mathbb{1} \cdot \sum_{k=0}^{N} \frac{(iz)^k}{k!} - \sum_{k=0}^{N} \frac{(ia)^k}{k!} := \mathbb{1} \cdot p_n(z) - p_n(a)$$

converges against p_∞, which is invertible. By lemma 1.4.7, the set $\mathbb{C} \setminus \sigma(\exp(ia))$ is open. Using the Neumann-series, one finds, that the set of invertible elements in $\tilde{A}$ is open in the norm topology. Thus, there has to be a finite $N \geq 0$, such that p_N is invertible. This yields:

$$p_N(z) \not\in \sigma(p_N(a)) \supset \sigma(p_N(a)) \ .$$

Using lemma 1.4.11 and $z \in \sigma_A(a)$, we find

$$p_N(a) \in p_N(\sigma(a)) = \sigma(p_N(a)) \ ,$$

which is a contradiction, such that the assumption $z \not\in \mathbb{R}$ was wrong. □

Lemma 2.2.6.
Let $\varphi \in \Gamma_A$, then there is a $\widetilde{\varphi} \in \Gamma_{\widetilde{A}}$, such that $\widetilde{\varphi}|_A \equiv \varphi$, defined by linear extension with $\widetilde{\varphi}(\mathbf{1}) = 1$. Also this extension is unique.

Proof 2.2.7.
Let φ act on $\pi(A)$, where $\pi \colon A \hookrightarrow \mathcal{L}(A)$ by $\varphi(\pi(a)) = a$. Define $\widetilde{\varphi} \in \Gamma_{\widetilde{A}}$ by

$$\widetilde{\varphi}(\pi(a) + z\mathbb{1}) = \varphi(a) + z \ .$$

Setting $z = 0$ shows that $\widetilde{\varphi}|_A \equiv \varphi$. It remains to show that $\widetilde{\varphi}$ is an algebra homomorphism. Since π is an algebra homomorphism we find:

$$\begin{aligned}\widetilde{\varphi}(c_1(\pi(a_1) + z_1\mathbb{1}) &+ c_1(\pi(a_1) + z_1\mathbb{1})) \\ &= \widetilde{\varphi}(\pi(c_1 a_1 + c_2 a_2) + (c_1 z_1 + c_2 z_2)\mathbb{1}) \\ &= \varphi(c_1 a_1 + c_2 a_2) + c_1 z_1 + c_2 z_2 \\ &= c_1(\varphi(a_1) + z_1) + c_2(\varphi(a_2) + z_2) \\ &= c_1\widetilde{\varphi}(\pi(a_1) + z_1\mathbb{1}) + c_2\widetilde{\varphi}(\pi(a_2) + z_2\mathbb{1})\end{aligned}$$

And from $(\pi(a) + z\mathbb{1})(\pi(b) + w\mathbb{1}) = \pi(ab + zb + wa) + zw\mathbb{1}$ it follows that:

$$\begin{aligned}\widetilde{\varphi}((\pi(a) + z\mathbb{1})(\pi(b) + w\mathbb{1})) &= \widetilde{\varphi}(\pi(ab + zb + wa) + zw\mathbb{1}) \\ &= \varphi(ab + zb + wa) + zw \\ &= \varphi(ab) + \varphi(zb) + \varphi(wa) + zw \\ &= (\varphi(a) + z)(\varphi(b) + w) \\ &= \widetilde{\varphi}(\pi(a) + z\mathbb{1})\widetilde{\varphi}(\pi(b) + w\mathbb{1}) \ .\end{aligned}$$

For uniqueness assume $\chi \in \Gamma_{\widetilde{A}}$, such that $\chi|A \equiv \varphi$. Since it is a homomorphism it has to hold that $\chi(1) = 1$, but then $\chi = \widetilde{\varphi}$. □

Remark 2.2.8.
In the last proof we would not have needed the explicit construction of the unitalization, as the theorem would have worked as well with $a + z\mathbf{1}$. However, it does not hurt to see the fallback method in action.

Theorem 2.2.9 (**Gelfand-Naimark theorem**).

Let A be a commutative C^*-algebra, then the Gelfand transformation

$$\Gamma\colon A \longrightarrow C_0(\Gamma_A) \ , \ \Gamma(a)(\varphi) := \varphi(a)$$

is a $*$-isomorphism that is an isometry w.r.t. the supremum norm $\|\cdot\|_\infty$.

Proof 2.2.10.
First we need to show that Γ is a $*$-morphism. The algebra structure on $C_0(\Gamma_A)$ is defined point wise, i.e. $(fg)(\varphi) = f(\varphi)\cdot g(\varphi)$. Since this defines a commutative algebra, the $*$ map is defined by $\overline{f}$. Linearity and multiplicativity follow from the fact that $\varphi \in \Gamma_A$ is an algebra homomorphism:

$$\begin{aligned}\Gamma(z_1a_1 + z_2a_2)(\varphi) &= \varphi(z_1a_1 + z_2a_2) = z_1\varphi(a_1) + z_2\varphi(a_2)\\ &= z_1\Gamma(a_1)(\varphi) + z_2\Gamma(a_2)(\varphi) \ ,\end{aligned}$$

$$\Gamma(ab)(\varphi) = \varphi(ab) = \varphi(a)\cdot\varphi(b) = \Gamma(a)(\varphi)\cdot\Gamma(b)(\varphi) \ .$$

The last thing to be checked for Γ being a $*$-morphism is $\Gamma(a^*) = \Gamma(a)^*$.

$$\varphi(a^*) = \Gamma(a^*)(\varphi) \overset{!}{=} \Gamma(a)^*(\varphi) = \overline{\Gamma(a)(\varphi)} = \overline{\varphi(a)} \ ,$$

thus $\varphi(a^*) = \overline{\varphi(a)}$ needs to be proven.

Let $\varphi \in \Gamma_A$ and $a \in A$ and use the extension $\widetilde{\varphi}$ from lemma 2.2.6 . Then $\widetilde{\varphi}(\varphi(a)\cdot \mathbb{1} - a) = \varphi(a)\cdot 1 - \varphi(a) = 0$ and thus $\varphi(a)\cdot\mathbb{1} - a \in \mathrm{Ker}(\widetilde{\varphi})$. Since $\mathrm{Ker}(\widetilde{\varphi})$ is a maximal ideal (see theorem 1.4.27), $\varphi(a)\cdot\mathbb{1} - a \in \mathrm{Ker}(\widetilde{\varphi})$ is not invertible (otherwise $\mathbb{1} \in \mathrm{Ker}(\widetilde{\varphi})$). Hence $\varphi(a) \in \sigma(a)$. In the case of self adjoint a, i.e. $a = a^*$, lemma 2.2.4 proves that $\varphi(a) \in \mathbb{R}$, such that

$$\varphi(a^*) = \varphi(a) = \overline{\varphi(a)} \ .$$

For general a, we use corollary 2.1.5 to find:

$$\begin{aligned}\varphi(a^*) &= \varphi(b^* - ic^*) = \varphi(b - ic) = \varphi(b) - i\varphi(c) = \overline{\varphi(b) + i\varphi(c)}\\ &= \overline{\varphi(b + ic)} = \overline{\varphi(a)} \ .\end{aligned}$$

This proves that Γ is a $*$-morphism.

That Γ is an isometry follows from corollary 2.2.2, since every element in a commutative algebra is normal:

$$\begin{aligned}\|a\| &= \rho(a) = \sup\{\|z\| \mid z \in \sigma(a)\} = \sup\{|\varphi(a)| \mid \varphi \in \Gamma_A\}\\ &= \|\Gamma(a)\|_\infty \ .\end{aligned}$$

The image of Γ is a subset of $C_0(\Gamma_A)$, i.e. $\Gamma(A) \subset C_0(\Gamma_A)$. Isometries map closed sets to closed sets, such that $\Gamma(A)$ is closed in $C_0(\Gamma_A)$. Furthermore, by definition $\Gamma(A)$ separates the points of Γ_A (and Γ is injective). By theorem 1.5.2, this proves surjectivity. □

Corollary 2.2.11.
It holds that $\overline{\varphi(a)} = \varphi(a^*)$

Proof 2.2.12.
This is also proven in the proof of theorem 2.2.9. □

Corollary 2.2.13.
Let A be a commutative C^*-algebra, then $\sigma(a) = \{\varphi(a) \mid \varphi \in \Gamma_A\}$.

Proof 2.2.14.
In the proof of theorem 2.2.9, we have already seen that $\varphi(a) \in \sigma(a)$. For the opposite direction we follow [Con97, proof of Thmeorem VII 8.6.].

Let $z \in \sigma(a)$, i.e. $z\mathbb{1} - a$ is not invertible. Then $I = (z\mathbb{1} - a)A$ is a proper ideal (because of commutativity). Let M be a maximal ideal of A, such that $I \subset M$ (existence because of Zorn's lemma). By theorem 1.4.27 ii), there is a $\widetilde{\varphi} \in \Gamma_{\widetilde{A}}$, such that $M = \mathrm{Ker}(\widetilde{\varphi})$. It follows that

$$0 = \widetilde{\varphi}(z\mathbb{1} - a) = z - \widetilde{\varphi}|_A(a) \qquad \Rightarrow \qquad z = \widetilde{\varphi}|_A(a) \ ,$$

where $\widetilde{\varphi}|_A$ is the restriction to A. But For the restriction to A it holds that $\chi \equiv \widetilde{\varphi}|_A \in \Gamma_A$, i.e. $\exists\ \chi \in \Gamma_A \colon z = \chi(a)$. □

2.3 Funtional calculus

To introduce the functional calculaus in C^*-algebras we will also follow parts of [Con97, VIII(2)].

Theorem 2.3.1.
Let A be a unital C^*-algebra and $B \subset A$ be a C^*-subalgebra that also contains the unit element. For $a \in B$ it holds that $\sigma_A(a) = \sigma_B(a)$.

The restrictions made in the theorem are only to simplify the notation in the proof. In fact:

Corollary 2.3.2.
Since the spectrum for non-unital C^*-algebras is defined with respect to $\widetilde{A}$ and $\widetilde{B}$, the previous theorem still holds, as long as $\mathbb{1} \in \widetilde{B}$.

Proof 2.3.3.
As with theorem 1.4.9, the inclusion $\sigma_A(a) \subset \sigma_B(a)$ is immediate. For the opposite

inclusion we define the C^*-subalgebra $C = C^*(a, \mathbb{1})$ that is generated by $a, \mathbb{1} \in A$. It follows that $C \subset B \subset C$ and $\sigma_A(a) \subset \sigma_B(a) \subset \sigma_C(a)$.

Let a be self adjoint first. Then C is commutative and by lemma 2.2.4 $\sigma_C(a) \subset \mathbb{R}$. In the topology of $\mathbb{C}$ it holds that $\sigma_C(a) = \partial\sigma_C(a)$ for that reason. From theorem 1.4.9 it follows that

$$\sigma_C(a) = \partial\sigma_C(a) \subset \partial\sigma_A(a) \subset \sigma_A(a) \qquad \Rightarrow \qquad \sigma_B(a) \subset \sigma_A(a)\ .$$

For a general b we show that if $b \in C$ is invertible in A it is also invertible in C, leading to $\sigma_C(b) \subset \sigma_A(b)$. So assume $b^{-1} \in A$, then $(b^*)^{-1} \in A$. Also $(b^*b)^{-1}b^* = b^{-1}(b^*)^{-1}b^* = b^{-1}$, such that it suffices to show that $(b^*b)^{-1} \in C$. Now, $b^*b \in C$ is self adjoint such that $\sigma_C(b^*b) \subset_A (b^*b)$ from the previous reasoning. Since $(b^*b)^{-1} \in A$, it holds that $0 \not\in \sigma_A(b^*b)$ and thus $0 \not\in \sigma_C(b^*b)$. But then b^*b is invertible in C, what we intended to show. □

Lemma 2.3.4.
Let A be a C^*-algebra and let $B = C^*(a)$ be the C^*-subalgebra of A generated by a. For every $z \in \sigma_A(a)$ there is a unique $\varphi_z \in \Gamma_{\widetilde{B}}$ with $\varphi_z(a) = z$. The map

$$j\colon \sigma_A(a) \longrightarrow \Gamma_{\widetilde{B}}\ , \qquad z \longmapsto \varphi_z$$

is a homeomorphism.

Proof 2.3.5.

1) The existence of φ_z is a consequence of theorem 2.3.1 and corollary 2.2.13. Assume now $\varphi_z, \chi_z \in \Gamma_{\widetilde{B}}$ with $\chi_z(a) = z = \varphi_z(a)$. Since $\widetilde{B}$ is generated by a, a^* and $\mathbb{1}$ and since φ_z and χ_z are Banach algebra homomorphisms, it follows that $\chi_z(b) = \varphi_z(b)$ for all $b \in \widetilde{B}$. Hence $\chi_z \equiv \varphi_z$, proving the uniqueness. This also shows injectivity.

2) Let $\varphi \in \Gamma_{\widetilde{B}}$ and choose $z := \varphi(a)$. By corollary 2.2.13 it holds that $z \in \sigma_A$. By the uniqueness it holds that $\varphi = \varphi_z$, such that $j(z) = \varphi_z = \varphi$, proving surjectivity.

3) By bijectivity, the map $j^{-1}\colon \varphi \mapsto \varphi(a) := z$ exists. As restriction of the map $a \mapsto \langle \cdot, a\rangle$ this shows that j^{-1} is continuous.

4) It remains to show that j is continuous. Let $U \subset \Gamma_{\widetilde{B}}$ be open, i.e. $\Gamma_{\widetilde{B}} \setminus U$ is closed. Since $\widetilde{B}$ is unital, $\Gamma_{\widetilde{B}}$ is compact, and thus $\Gamma_{\widetilde{B}} \setminus U$ is also compact. Since j^{-1} is continuous and hence maps compact sets to compact sets,

$$j^{-1}(\Gamma_{\widetilde{B}} \setminus U) = j^{-1}(\Gamma_{\widetilde{B}}) \setminus j^{-1}(U) = \sigma_A(a) \setminus j^{-1}(U)$$

is compact. Furthermore, $\sigma_A(a)$ is closed by the proof of lemma 1.4.7, such that

$$j^{-1}(U) = \sigma_A(a) \setminus (\sigma_A(a) \setminus j^{-1}(U)$$

is open. Hence j is continuous.

□

Corollary 2.3.6.
For every $z \in \sigma_A(a) \setminus \{0\}$ there is a unique $\varphi_z \in \Gamma_B$ with $\varphi_z(a) = z$.

Proof 2.3.7.
Existence and uniqueness are shown exactly as before. □

Corollary 2.3.8.
The spaces $\sigma_A(a) \setminus \{0\}$ and Γ_B are homeomorphic by the restriction $j|_{\sigma_A(a)\setminus\{0\}}$.

Proof 2.3.9.
Surjectiviy follows as before, except for $z = 0$, which is excluded. Let $\varphi \in \Gamma_B$ and set $z = \varphi(a)$. If $z = 0$, then $\varphi \equiv 0$, since $B = C^*(a)$, which is a contradiction to $\varphi \in \Gamma_B$. □

Lemma 2.3.10.
Let X and Y be homeomorphic spaces, then $C(X) \cong C(Y)$ and $C_0(X) \cong C_0(Y)$ as C^*-algebras.

Proof 2.3.11.
Let $j\colon X \to Y$ be a homeomorphism. Define the map $j^\#\colon C(Y) \to C(X)$ by $f \mapsto f \circ j$. The inverse $(j^\#)^{-1}$ is $(j^{-1})^\#$. It remains to show, that $j^\#$ is a $*$-morphism.

$$j^\#(\alpha f + \beta g) = (\alpha f + \beta g) \circ j = \alpha \cdot (f \circ j) + \beta \cdot (g \circ j) = \alpha j^\#(f) + \beta j^\#(g) \ ,$$

$$j^\#(fg) = (fg) \circ j = (f \circ j)(g \circ j) = j^\#(f) j^\#(g) \ ,$$
$$j^\#(\overline{f})(x) = \overline{f}(j(x)) = \overline{f(j(x))} = \overline{f \circ j}(x) = \overline{j^\#(f)}(x) \ .$$

For $C_0(X) \cong C_0(Y)$, we only need to check that $j^\#(f) \in C_0(X)$ for $f \in C_0(Y)$, i.e.

$$\{x \in X \mid |f(j(x))| > \varepsilon\} \qquad \text{is compact in } X \ .$$

It holds that

$$\begin{aligned} j^{-1}(\{y \in Y \mid |f(y)| > \varepsilon\}) &= \{j^{-1}(y) \mid y \in Y,\ |f(y)| > \varepsilon\} \\ &= \{x \in X \mid |f(j(x))| > \varepsilon\} \ . \end{aligned}$$

Since $\{y \in Y \mid |f(y)| > \varepsilon\}$ is compact and j^{-1} continuous,
$\{x \in X \mid |f(j(x))| > \varepsilon\}$ is compact in X. □

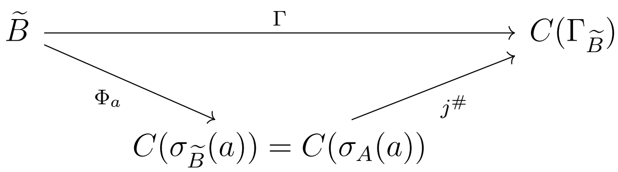
Lemma 2.3.12.
Let A be a C^*-algebra and $a \in A$ normal. Let $B = C^*(a)$ be the C^*-subalgebra generated by a. Then the map

$$\Phi_a \colon \widetilde{B} \longrightarrow C(\sigma_A(a)) \ , \quad \Phi_a(b)(z) := \varphi_z(b)$$

is a $*$-isomorphism, that restricts to a $*$-isomorphism

$$B \longrightarrow C_0(\sigma_A(a) \setminus \{0\}) \ .$$

Proof 2.3.13.
Since a is normal, $B = C^*(a)$ and $\widetilde{B} = \widetilde{C^*(a)}$ are commutative. From theorem 2.3.1 it follows that $\sigma_A a := \sigma_{\widetilde{A}}(a) = \sigma_{\widetilde{B}}(a)$. Consider the following diagram:

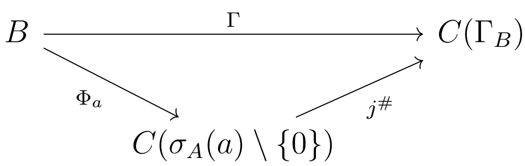


Since Γ is a $*$-isomorphism by theorem 2.2.9 and $(j^{-1})^\#$ is a $*$-isomorphism by lemma 2.3.4 and 2.3.10, all that is left to show is, that the diagram commutes.

$$\begin{aligned}[(j^\# \circ \Phi_a)(b)](\varphi_z) &= [j^\#(\Phi_a(b))](\varphi_z) = \Phi_a(b)(j(\varphi_z)) \\ &= \Phi_a(b)(z) = \varphi_z(b) \ ,\end{aligned}$$

$$\Gamma(b)(\varphi_z) = \varphi_z(b) = [(j^\# \circ \Phi_a)(b)](\varphi_z) \ .$$

For the restriction, use again theorem 2.2.9, corollary 2.3.6, lemma 2.3.10 and consider



$\square$

Definition 2.3.14.
Let A be a C^*-algebra and $a \in A$ normal. The **functional calculus** of a is defined by $\Phi_a^{-1} \colon C(\sigma_A(a)) \to C^*(a, \mathbb{1})$. A common notation is

$$f(a) := \Phi_a^{-1}(f) \ .$$

The notation of the functional calculus will also be used for

$$\Phi_a^{-1} \colon C_0(\sigma_A(a) \setminus \{0\}) \to C^*(a) \ .$$

Remark 2.3.15.
It is possible for a non-unital C^*-algebra to have a unital C^*-sub algebra B, as $\mathbb{1}_B b = b\mathbb{1}_B$ needs to hold only for $b \in B$. An example for this is the unitalization of a unital C^*-algebra. Then $\mathbb{1}_A \neq \mathbb{1}_{\widetilde{A}}$ in general. This does not violate the uniqueness of the unit element, as $\mathbb{1}_A$ is not a unit on the whole of $\widetilde{A}$.
Thus, even for a non-unital C^*-algebra the function $f(z) = 1$ can be in $C_0(\sigma(a) \setminus 0)$ and correspond to $f(a) = \mathbb{1}$ where $\mathbb{1} = \mathbb{1}_{C^*(a)}$.

Corollary 2.3.16.
Let $a \in A$ be normal, $f \in C(\sigma(a))$ and $g \in C(f(\sigma(a)))$, then :

$$\sigma(f(a)) = f(\sigma(a)) \qquad \textit{and} \qquad g(f(a)) = (g \circ f)(a) \ .$$

Proof 2.3.17.
For this proof, we follow [Mur90, p. 43].

Let $\varphi \in \Gamma_{C^*(a,\mathbb{1})}$. We want to show that $\varphi(f(a)) = f(\varphi(a))$ for all $f \in C(\sigma(a))$. By lemma 2.3.4 there is a $z \in \sigma(b)$ with $\varphi = \varphi_z$ w.r.t. a. It follows that:

$$\varphi_z(f(a)) = \Phi_b(f(a))(z) = \Phi_b(\Phi_b^{-1}(f))(z) = f(z) = f(\varphi_z(a)) \ .$$

But then by corollary 2.2.13:

$$\begin{aligned} \sigma(f(a)) &= \{\varphi(f(a)) \mid \varphi \in \Gamma_{C^*(a,\mathbb{1})}\} = \{f(\varphi(a)) \mid \varphi \in \Gamma_{C^*(a,\mathbb{1})}\} \\ &= f(\{\varphi(a) \mid \varphi \in \Gamma_{C^*(a,\mathbb{1})}\}) = f(\sigma(a)) \ . \end{aligned}$$

It holds that $C^*(f(a), \mathbb{1}) \subset C^*(a, \mathbb{1})$ and thus $g \in C(f(\sigma(a))) = C(\sigma(f(a)))$. The restriction obviously satisfies $\varphi|_{C^*(f(a),\mathbb{1})} \in \Gamma_{C^*(f(a),\mathbb{1})}$ for $\varphi \in \Gamma_{C^*(a,\mathbb{1})}$. Hence:

$$\varphi((g \circ f)(a)) = g(f(\varphi(a))) = g(\varphi(f(a))) = \varphi(g(f(a)))$$

$$\Rightarrow \qquad (g \circ f)(a) = g(f(a)) \ .$$

□

Corollary 2.3.18.
Let A and B be C^*-algebras and $\phi\colon A \to B$ be a $*$-morphism, then $\sigma_B(\phi(a)) \subset \sigma_A(a)$ and for all $f \in C(\sigma_A(a))$ it holds that:

$$f(\phi(a)) = \phi(f(a)) \ .$$

Proof 2.3.19.
In the same way we could extend $\varphi \in \Gamma_A$ to $\widetilde{\varphi} \in \Gamma_{\widetilde{A}}$, see lemma 2.2.6, we can extend ϕ to $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$. For ease of notation we identify ϕ and $\widetilde{\phi}$ by $\phi(\mathbb{1}) = \mathbb{1}$.

Noticing that $\phi(A) \subset B$, it follows that $\sigma_B(\phi(a)) \subset C_{\phi(A)}(\phi(a))$. Assume now that $z\mathbb{1}a$ is invertible in A, then there is a $c \in A$ such that

$$c(z\mathbb{1} - a) = (z\mathbb{1} - a)c = \mathbb{1}$$

$$\begin{aligned} \Rightarrow \quad \phi(c)(z\mathbb{1} - \phi(a)) &= \phi(c)\phi(z\mathbb{1} - a) = \phi(c(z\mathbb{1} - a)) = \phi(1) \\ &= \mathbb{1} = \ldots = (z\mathbb{1} - \phi(a))\phi(c) \ . \end{aligned}$$

This means that $(z\mathbb{1} - \phi(a))$ is invertible in $\phi(a)$, such that

$$\sigma_B(\phi(a)) \subset C_{\phi(A)}(\phi(a)) \subset \sigma_A(a) \ .$$

This ensures that f is defied on $\sigma_B(\phi(a))$. As seen in the proof of corollary 2.3.16, for all $\varphi \in \Gamma_{C^*(a,\mathbb{1})}$ it holds that $f(\varphi(a)) = \varphi(f(a))$. Since $\varphi \circ \phi \in \Gamma_{C^*(\phi(a),\mathbb{1})}$ it follows that:

$$\begin{aligned} \varphi(\phi(f(a))) = (\varphi \circ \phi)(f(a)) = f((\varphi \circ \phi)(a) = f(\varphi(\phi(a))) = \varphi(f(\phi(a))) \\ \Rightarrow \quad f(\phi(a)) = \phi(f(a)) \ . \end{aligned}$$

□

We conclude this section with a useful result from [RLL00, lemma 1.2.5]

Lemma 2.3.20.
Let K be a non-empty compact subset of $\mathbb{R}$, and let $f\colon K \to \mathbb{C}$ be a continuous function. If $\Omega_K \subset A$ is the set of self adjoint elements of a C^*-algebra A with spectrum in K, then the induced map

$$f\colon \Omega_K \longrightarrow A \ , \quad a \longmapsto f(a)$$

is continuous.

Proof 2.3.21.
Since multiplication is continuous for elements in A, polynomials induce continuous maps $p\colon A \to A$. A result of the Stone-Weierstrass theorem is, that for $\varepsilon > 0$, there exists a complex polynomial, such that

$$|f(z) - p(z)| \leq \frac{\varepsilon}{3} \quad \forall\, z \in K \ .$$

Let $\delta > 0$, such that

$$\|a - a_0\| \leq \frac{\varepsilon}{3} \quad \forall\, a \in B_\delta(a_0) \ .$$

With the sup-norm, it follows that

$$\|f(c) - p(c)\| = \|(f-p)(c)\| = \sup\{|(f-p)(z)| \mid z \in \sigma(c) \subset K\} \leq \frac{\varepsilon}{3} \ ,$$

for all $c \in \Omega_K$. Thus with the triangle inequality, it holds that $\|f(a) - f(a_0)\| \leq \varepsilon$ for all $a \in \Omega_K \cap B_\delta a_0$. □

2.4 Positive elements

In lemma 2.2.4 it was shown, that $\sigma(a) \subset \mathbb{R}$ for normal a. This will be used to define positive elements.

Definition 2.4.1.
An element $a \in A$ is called **positive** if it is normal and $\sigma(a) \subset \mathbb{R}_{\geq 0}$. For a subset $B \subset A$, the set B_+ denotes the set of positive elements in B.

It is common to write $a \geq 0$ for positive elements. In that sense one writes $a \geq b$, if $a - b \geq 0$ etc.

Lemma 2.4.2.
Let $a \in A$ be normal. Then a is self adjoint if and only if $\sigma(a) \subset \mathbb{R}$

Proof 2.4.3.
Since Φ_a is a $*$-isomorphism, choose $f \in C(\sigma(a))$ such that $a^* = f(a)$. Then:

$$f(z) = \Phi_a(a^*)(z) = \varphi_z(a^*) = \overline{\varphi_z}(a) = \overline{\varphi_z(a)} = \overline{z} \ .$$

On $\mathbb{R}_{\geq 0} \supset \sigma(a)$ the function f restricts to $\mathrm{Id}_{\sigma(a)}$, such that $a^* = f(a) = \Phi_a^{-1}(\mathrm{Id}_{\sigma(a)}) = a$.

The opposite direction is lemma 2.2.4. □

Example 2.4.4.
Let $a \in A$ be a positive element. Consider the function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f(r) = \sqrt{r}$ and set

$$a^{\frac{1}{2}} := f(a) = \sqrt{a} \ .$$

By lemma 2.4.2, a is self adjoint. Furthermore let $a^2 = g(a)$, then

$$g(z) = \Phi_a(a^2)_z = \varphi_z(a^2) = (\varphi_z(a))^2 = z^2 \ ,$$

and by corollary 2.3.16

$$(a^{\frac{1}{2}})^2 = g(f(a)) = (g \circ f)(a) = \mathrm{Id}(a) = a$$

since

$$\Phi_a(a)(z) = \varphi_z(a) = z = \mathrm{Id}(z) \ .$$

Lemma 2.4.5.
Let $a \in A$ be self adjoint. Then $a_+ = f(a)$ with $f(z) = \max(z, 0)$ and $a_- = g(a)$ with $g(z) = f(z) - \mathrm{Id}_{\mathbb{R}}$, such that

$$a_\pm \in A_+ \ , \qquad a = a_+ - a_- \quad \textit{and} \quad a_+ a_- = 0 \ .$$

Proof 2.4.6.
By corollary 2.3.16 $\sigma(a_\pm) \subset \mathbb{R}_{\geq 0}$. By lemma 2.4.2 $a_\pm$ are self adjoint, hence normal and thus $a_\pm \in A_+$. From $f(z) - g(z) = \mathrm{Id}_\mathbb{R}$ it follows that (see example 2.4.4)

$$\begin{aligned} a_+ - a_- &= f(a) - g(a) = \Phi_a^{-1}(f) - \Phi_a^{-1}(g) \\ &= \Phi_a^{-1}(f - g) = \Phi_a^{-1}(\mathrm{Id}_\mathbb{R}) = a \ . \end{aligned}$$

Similarly from $a_+a_- = a_+(a_+ - a) = a_+^2 - a_+a$, $(f(z))^2 - f(z) \cdot z = 0$ and $\Phi_a^{-1}(0) = 0$, it follows that $a_+a_- = 0$. □

Lemma 2.4.7.
Let $a \in A$, then the following claims are equivalent:

i) *a is a positive element.*

ii) *Let $A_{sa} \subset A$ denote the subsetset of self adjoint elements. Then $\exists\, b \in A_{sa}$, such that $a = b^2$.*

iii) *$a \in A_{sa}$ and $\|t \cdot \mathbb{1} - a\| \leq t$ for all $t \geq \|a\|$.*

iv) *$a \in A_{sa}$ and $\|t \cdot \mathbb{1} - a\| \leq t$ for one $t \geq \|a\|$.*

Proof 2.4.8.

i) ⇔ ii)
Being a positive element means that $\sigma(a) \subset \mathbb{R}_{\geq 0} \subset \mathbb{R}$, hence $a \in A_{sa}$ by lemma 2.4.2. From example 2.4.4 we know that $b = a^{\frac{1}{2}} \in A_{sa}$ and $b^2 = (a^{\frac{1}{2}})^2 = a$.

On the other hand, assume there is a $b \in A_{sa}$, such that $a = b^2$. Since $b \in A_{sa}$, it holds that $\sigma(a) \subset \mathbb{R}$ and thus by corollary 2.3.16

$$\sigma(a) = \sigma(b^2) = (\sigma(b))^2 \subset \mathbb{R}_{\geq 0} \ .$$

Finally since $a = b^2$, a is self adjoint and thus normal.

i) ⇒ iii)
With $t \cdot \mathbb{1} - a = f(a)$ it holds that

$$f(z) = \Phi_a(t \cdot \mathbb{1} - a)(z) = \varphi_z(t \cdot \mathbb{1} - a) = t - z \ .$$

Let $t \geq \|a\| = \rho(a) \geq 0$. Using corollary 2.2.2 and 2.3.16 it follows that

$$\begin{aligned} \|t \cdot \mathbb{1} - a\| &= \rho(t \cdot \mathbb{1} - a) = \sup |\sigma(t \cdot \mathbb{1} - a)| \\ &= \sup |t - \sigma(a)| = \sup\{|t - z| \mid z \in \sigma(a)\} \\ &= \sup\{t - |z| \mid z \in \sigma(a)\} \leq t \ . \end{aligned}$$

iii) $\Rightarrow$ iv)
This is obvious.

iv) $\Rightarrow$ i)
Let $z \in \sigma(a) \subset \mathbb{R}$ and $t \geq \|a\|$ such that $\|t \cdot \mathbb{1} - a\| \leq t$. It holds that $z - t \in \sigma(t\mathbb{1} - a)$, since

$$(z-t) \cdot \mathbb{1} - (t \cdot \mathbb{1} - a) = z \cdot \mathbb{1} - a$$

is not invertible for $z \in \sigma(a)$. Furthermore $z \leq |z| \leq \rho(a) = \|a\| \leq t$ such that

$$t - z = |t - z| \leq \rho(t \cdot \mathbb{1} - a) = \|t \cdot \mathbb{1} - a\| \leq t \ ,$$

hence $z \geq 0$.

□

For the next theorem, we define the term convex cone. Let $C \subset V$ be a subset of a $\mathbb{K}$ vector space. C is called **convex cone**, if

$$\forall\ x, y \in C\ , \forall\ \alpha, \beta \in \mathbb{R}_{>0}\ : \quad \alpha x + \beta y \in C\ .$$

Theorem 2.4.9.
The set A_+ is a closed convex cone in A_{sa}. It holds that $a \geq 0$ if and only if there is $b \in A$, such that $a = b^* b$.

Proof 2.4.10.
Corresponding to $f(z) = \alpha \cdot z$ for $\alpha \in \mathbb{R}_{\geq 0}$ is $f(a) = \alpha \cdot a$. By corollary 2.3.16 it holds that $\alpha a \in A_+$ if $a \in A_+$. To see that A_+ is a convex cone, it remains to show that $a + b \in A_+$ for $a, b \in A_+$. Let $a, b \in A_+$, $s \geq \|a\|$ and $t \geq \|b\|$, then by lemma 2.4.7:

$$\|(s+t) - (a+b)\| \leq \|s - a\| + \|t - b\| \leq s + t \qquad \Rightarrow \qquad a + b \in A_+\ .$$

It can be shown that A_{sa} by continuity of the map $*\colon A \to A$. Let (a_k) be a sequence in A_+ with $a_k \to a \in A_{sa}$. Take $t \geq \sup_k \|a_k\|$, then $t \geq \|a\|$ and

$$\|t\mathbb{1} - a\| = \lim_{k\to\infty} \|t\mathbb{1} - a_k\| \leq t\ ,$$

such that $a \in A_+$.
If $a \geq 0$, i.e. $a \in A_+$, then it holds that there is $b \in A_{sa} \subset A$ such that $a = b^2 = b^* b$, by lemma 2.4.7. For the opposite direction assume $a = b^* b$ for $b \in A$. Then $a \in A_{sa}$. By lemma 2.4.5 it holds that $a = a_+ - a_-$. With lemma 2.4.7 it follows that:

$$\begin{aligned}
(ba_-^{\frac{1}{2}})^*(ba_-^{\frac{1}{2}}) &= a_-^{\frac{1}{2}} b^* b a_-^{\frac{1}{2}} = a_-^{\frac{1}{2}} a a_-^{\frac{1}{2}} \\
&= a_-^{\frac{1}{2}}(a_+ - a_-)a_-^{\frac{1}{2}} = (a_- a_+^2 a_-)^{\frac{1}{2}} - (a_- a_-^2 a_-)^{\frac{1}{2}} \\
&= (a_- a_+)^{\frac{1}{2}}(a_+ a_-)^{\frac{1}{2}} - a_-^2 = -a_-^2 \in -A_+\ .
\end{aligned}$$

Using corollary 2.1.5 to define $x, y \in A_{sa}$ such that $ba_-^{\frac{1}{2}} = x + iy$ it follows that

$$x = \frac{1}{2}\left(ba_-^{\frac{1}{2}} + a_-^{\frac{1}{2}}\right) \qquad \text{and} \qquad y = \frac{1}{2i}\left(ba_-^{\frac{1}{2}} - a_-^{\frac{1}{2}}b\right) .$$

A lengthy calculation shows that

$$(ba_-^{\frac{1}{2}})(ba_-^{\frac{1}{2}})^* = x^2 + y^2 + i(xy - yx) = \ldots = 2(a^2 + b^2) - a^2 \in A_+ ,$$

since A_+ is a convex cone. Thus $\sigma((ba_-^{\frac{1}{2}})(ba_-^{\frac{1}{2}})^*) \subset \mathbb{R}_{\geq 0}$, while $\sigma((ba_-^{\frac{1}{2}})^*(ba_-^{\frac{1}{2}})) \subset \mathbb{R}_{\leq 0}$. By corollary 1.4.4 it follows that:

$$\sigma(-a_-^2) \cup \{0\} = \sigma((ba_-^{\frac{1}{2}})(ba_-^{\frac{1}{2}})^*) \cup \{0\} = \{0\} .$$

Since a_- is normal, so is $-a_-^2$ and form $0 = \rho(-a_-^2) = \|a_-^2\|$ it also follows that $a_- = 0$ since $0 = \|a_-^2\| = \|a_-^* a_-\| = \|a_-\|^2$. Hence $a = a_+ \in A_+$. □

Lemma 2.4.11.
Let $a \in A$ be self adjoint, then in $\widetilde{A}$ it holds that $a \leq \|a\|\mathbb{1}$.

Proof 2.4.12.
As in the proof (i) $\Rightarrow$ iii)) of lemma 2.4.7 consider $\|a\|\mathbb{1} - a = f(a)$. Using corollary 2.3.16 it follows that

$$\sigma(\|a\|\mathbb{1} - a) = \sigma(f(a)) = f(\sigma(a)) = \|a\| - \sigma(a) \subset \mathbb{R}_{\geq 0} ,$$

since in general $\|a\| \geq \rho(a) \geq z$ for all $z \in \sigma(a)$. This shows that $\|a\|\mathbb{1} - a \geq 0$, which is by definition equivalent to $a \leq \|a\|\mathbb{1}$. □

Corollary 2.4.13.
Let $a \leq b$ and $x \in A$, then $x^*ax \leq x^*bx$. If $a \geq 0$, then $b \leq \|b\| \cdot \mathbb{1}$ in $\widetilde{A}$ and $\|a\| \leq \|b\|$.

Proof 2.4.14.
By definition of the expression $a \leq b$, the element $b - a$ is positive. Theorem 2.4.9 shows that there is a $c \in A$, such that $b - a = c^*c$. Hence:

$$x^*bx - x^*ax = x^*(b - a)x = x^*c^*cx = (cx)^*(cx) \geq 0$$

$$\Leftrightarrow \qquad x^*ax \leq x^*bx .$$

Since $b \geq 0$ it is self adjoint and by lemma 2.4.11 $b \leq \|b\|\mathbb{1}$, such that $a \leq b \leq \|b\|$.
Assume now that $a \geq 0$. As in the proof (iv) $\Rightarrow$ i)) of lemma 2.4.7 it holds that $\|b\| - z \in \sigma(\|b\|\mathbb{1} - a)$ for $z \in \sigma(a)$. From $a \leq \|b\|\mathbb{1}$ it follows that $\sigma(\|b\|\mathbb{1} - a) \subset \mathbb{R}_{\geq 0}$ and thus $\|b\| - z \geq 0$. Hence

$$\|b\| - \|a\| = \inf\{\|b\| - z \mid z \in \sigma(a)\} \geq 0 .$$

□

Corollary 2.4.15.
Let A be unital and $a, b \in A$ be invertible with $0 \leq a \leq b$, then it holds that $0 \leq b^{-1} \leq a^{-1}$.

Proof 2.4.16.
From corollary 2.3.16 it immediately follows that $b^{-1} \geq 0$. Using the last corollary 2.4.13 we find:

$$b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \leq b^{-\frac{1}{2}} b b^{-\frac{1}{2}} = \mathbb{1}$$

and also since positive elements are self adjoint:

$$\|a^{\frac{1}{2}} b^{-\frac{1}{2}}\| = \left\| \left(a^{\frac{1}{2}} b^{-\frac{1}{2}}\right)^* a^{\frac{1}{2}} b^{-\frac{1}{2}} \right\|^{\frac{1}{2}} = \|b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\|^{\frac{1}{2}} \leq \|\mathbb{1}\| = 1 \ .$$

Furthermore since $a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}$ is self adjoint (for a and b are), it holds that $a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \leq \|a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\| \mathbb{1}$, such that:

$$\begin{aligned} a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \leq & \|a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\| \mathbb{1} = \|a^{\frac{1}{2}} b^{-\frac{1}{2}} b^{-\frac{1}{2}} a^{\frac{1}{2}}\| \mathbb{1} \\ & = \left\| a^{\frac{1}{2}} b^{-\frac{1}{2}} \left(a^{\frac{1}{2}} b^{-\frac{1}{2}} \right) \right\| \mathbb{1} = \|a^{\frac{1}{2}} b^{-\frac{1}{2}}\|^2 \mathbb{1} \leq \mathbb{1} \ . \end{aligned}$$

Hence we have:

$$b^{-1} = a^{-\frac{1}{2}} \left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \right) a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}} \mathbb{1} a^{-\frac{1}{2}} = a^{-1} \ .$$

□

Remark 2.4.17.
In fact, even without $0 \leq a$ it holds that $a \leq b \ \Rightarrow \ b^{-1} \leq a^{-1}$, as long as a and b are self adjoint.

Definition 2.4.18.
Let $f \colon I \to \mathbb{R}$ be a continuous function on an interval $I \subset \mathbb{R}$. Then f is called **operator monotone increasing**, if $f(a) \leq f(b)$ whenever $a \leq b$ for normal $a, b \in A$ with $\sigma(a) \cup \sigma(b) \subset I$. In the same way one defines **operator monotone decreasing**.

Example 2.4.19.
For $\alpha >$ consider the function

$$f_\alpha \colon (-\tfrac{1}{\alpha}, \infty) \longrightarrow \mathbb{R} \ , \qquad t \longmapsto \frac{t}{1 + \alpha t} = \tfrac{1}{\alpha}(1 - (1 + \alpha t)^{-1}) \ .$$

This function leads to $f(a) = \frac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha a)^{-1})$:

$$f(t) = \Phi_a(\tfrac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha a)^{-1}))(t) = \varphi_t(\tfrac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha a)^{-1}))$$

$$= \tfrac{1}{\alpha}(\varphi_t(\mathbb{1}) - (\varphi_t(\mathbb{1} - \alpha a)^{-1}) = \tfrac{1}{\alpha}(1 - (1 + \alpha t)^{-1}) \ .$$

It also holds that $\frac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha t)^{-1}) = a(\mathbb{1} + \alpha a)^{-1}$ as can be seen as follows.

$$\begin{aligned} f(t) &= \Phi_a(a(\mathbb{1} + \alpha a)^{-1})(t) = \varphi_t(a(\mathbb{1} + \alpha a)^{-1}) = \varphi_t(a)\varphi_t((\mathbb{1} + \alpha a)^{-1}) \\ &= t \cdot (\varphi_t(\mathbb{1} + \alpha a))^{-1} = \frac{t}{1 + \alpha t} \ . \end{aligned}$$

From $\sigma(a) \subset \mathbb{R}$ and $\sigma(b) \subset \mathbb{R}$ by definition of operator monotony and from lemma 2.4.2 it follows that a and b have to be self adjoint. Assume $a \leq b$, then by corollary 2.4.15 (and its remark) it holds that $b^{-1} \leq a^{-1}$. It follows that:

$$a \leq b \quad \Leftrightarrow \quad b - a \geq 0 \ ,$$

$$\begin{aligned} (\mathbb{1} - a) - (\mathbb{1} - b) = b - a \geq 0 \qquad &\Leftrightarrow \qquad \mathbb{1} - b \leq \mathbb{1} - a \\ \Rightarrow \qquad &(\mathbb{1} - a)^{-1} \leq (\mathbb{1} - b)^{-1} \ . \end{aligned}$$

Also, for $x \leq y$ since $(1 + y) - (1 + x) = y - x \geq 0$ it follows that $\mathbb{1} + y \leq \mathbb{1} + x$, such that:

$$f(a) = \tfrac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha a)^{-1}) \leq \tfrac{1}{\alpha}(\mathbb{1} - (\mathbb{1} + \alpha b)^{-1}) = f(b) \ .$$

Thus the functions f_α are operator monotone increasing.

With methods from analysis, the following properties of the functions f_α can be shown:

i) $f_\alpha(t) < \min(t, \frac{1}{\alpha})$.

ii) $\lim_{\alpha \searrow 0} f_\alpha(t) = t$ uniformly for t in compact subset of $\mathbb{R}$.

iii) $f_\alpha \geq f_\beta$ for all $\alpha \leq \beta$.

iv) $f_\alpha \circ f_\beta = f_{\alpha+\beta}$ on $(-\frac{1}{\alpha+\beta}, \infty)$.

v) $\lim_{t \to \infty} \alpha f_\alpha(t) = 1$.

Theorem 2.4.20.
Let $0 < \beta \leq 1$, then $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}, t \mapsto t^\beta$ is operator monotone increasing.

Proof 2.4.21.
Direct calculations show that

$$\int_0^\infty f_\alpha(t)\alpha^{-\beta} \, d\alpha = Ct^\beta \ , \quad C := \int_0^\infty (1 + \alpha)^{-1}\alpha^\beta \, d\alpha > 0 \ .$$

Let $0 \leq a \leq b$ in A and $\varepsilon > 0$. There are $n, m \geq 1$, such that with $\alpha_k := \frac{kn}{m}$ it holds that

$$\left| t^\beta - \frac{n}{Cm} \sum_{k=0}^{m} f_{\alpha_k}(t)\alpha_k^{-\beta} \right| \leq \varepsilon \quad \forall \ t \in [0, \|b\|] \ .$$

Since $f_{\alpha_k}(b) - f_{\alpha_k}(a) \geq 0$ and since A_+ is a convex cone by theorem 2.4.9, it holds that

$$c := \frac{n}{Cm} \sum_{k=0}^{m} (f_{\alpha_k}(b) - f_{\alpha_k}(a)) \alpha_k^{-\beta} \geq 0 \ .$$

Since the functional calculus is an isometry, it also holds that

$$\|b^\beta - a^\beta - c\| \leq 2\varepsilon \ .$$

With corollary 2.4.13 $b^\beta - a^\beta - c \leq \|b^\beta - a^\beta - c\|\mathbb{1}$ we find

$$-2\varepsilon\mathbb{1} \leq -\|b^\beta - a^\beta - c\|\mathbb{1} \leq b^\beta - a^\beta - c$$

and since A_+ is a convex cone,

$$b^\beta - a^\beta + 2\varepsilon\mathbb{1} = (b^\beta - a^\beta - c + 2\varepsilon\mathbb{1}) + c \geq 0 \ .$$

Since A_+ is closed (also by theorem 2.4.9) and ε was arbitrary, the claim follows. □

2.5 Approximate units

Before we define approximate units we recall the basics of (topological) nets.

Definition 2.5.1.
A set $\Lambda \neq \emptyset$ is called **directed set**, if there is a preorder relation $\leq$, i.e. $\leq$ is reflexive and transitive, such that for any two $\alpha, \beta \in \Lambda$ there is an upper bound $\mu \in \Lambda$:

$$\alpha \leq \mu \qquad \text{and} \qquad \beta \leq \mu \ .$$

A **net** on a topological space X is a family $(x_\lambda)_{\lambda\in\Lambda}$ with $x_\lambda \in X$ and a directed set Λ.

Recalling sequences, convergence of a net is defined as follows:

Definition 2.5.2.
A net $(x_\lambda)_{\lambda\in\Lambda}$ is called **converging against** $x \in X$, i.e. $\lim_{\lambda\to\infty} x_\lambda$, if for every neighborhood U of x there is a $\lambda_0 \in \Lambda$, such that $x_\lambda \in U$ for all $\lambda_0 \leq \lambda$.

With the help of nets we can define approximate units:

Definition 2.5.3.
A net $(u_\lambda)_{\lambda\in\Lambda}$ in A_+, such that $\|u_\lambda\| \leq 1$ and $u_\lambda \leq u_\mu$ for all $\lambda \leq \mu$ is called **approximate unit**, if for all $a \in A$ it holds that $a = \lim_{\lambda\to\infty} a u_\lambda$.

For an approximate unit it follows that

$$\|(\mathbb{1} - u_\lambda)a\| = \|a^*(\mathbb{1} - u_\lambda)\| \longrightarrow 0 \qquad \forall\ a \in A,$$

such that it also holds that $a = \lim_{\lambda\to\infty} u_\lambda a$.

Remark 2.5.4 (Usage of $\mathbb{1}$ in non-unital C^*-algebras)**.**
There have been numerous instances so far, that we have not explicitly stated that a C^*-algebra is unital, but written $\mathbb{1}$. There are two reasons, why this was possible. The first is, that we were interested in properties concerning the spectrum, in which case one passes to the unitalized algebra anyway. The second case has just happened. Since $a - u_\lambda \in A$ so is $(\mathbb{1} - u_\lambda)a$, although $\mathbb{1}$ is from the unitalization.

Remark 2.5.5 (approximate units and unitalization)**.**
In the definition of approximate units, the convergence of the net is not demanded. Only the limit $\lim\limits_{\lambda\in\Lambda} au_\lambda$ is specified. Furthermore, an approximate unit of A is not necessarily an approximate unit of $\widetilde{A}$, since $\lim\limits_{\lambda\in\Lambda} \pi(\lambda)\mathbb{1}$ need not converge, or put differently, $\lim\limits_{\lambda\in\Lambda} \tilde{a}u_\lambda$ need not converge for $\tilde{a} \in \widetilde{A} \setminus A$.
If however A is unital and $\mathbb{1}$ is its unit, then $\lim\limits_{\lambda\in\Lambda} u_\lambda = \mathbb{1}$.

Theorem 2.5.6.
Every C^*-algebra has an approximate unit.

Proof 2.5.7.
Define the set

$$\Lambda := \{x \in A_+ \mid \|x\| < 1\}$$

with the preorder $\leq$ from A_+. The first thing to show is, that Λ is a directed set. Let $x, y \in \Lambda$. By corollary 2.3.16 one can see that $a := x(\mathbb{1} - x)^{-1} \in A_+$ and $b := y(\mathbb{1} - y)^{-1} \in A_+$. As preparation for the next step, we calculate:

$$\mathbb{1} = \mathbb{1} - x + x = (\mathbb{1} - x) + (\mathbb{1} - x)x(\mathbb{1} - x)^{-\mathbb{1}}$$

$$\begin{aligned} \Leftrightarrow \quad (\mathbb{1} - x)^{-\mathbb{1}} &= \mathbb{1} + x(\mathbb{1} - x)^{-\mathbb{1}} \\ \Leftrightarrow \quad \mathbb{1} &= (\mathbb{1} - x)^{-1}(\mathbb{1} + x(\mathbb{1} - x)^{-1})^{-1} \\ \Leftrightarrow \quad x &= x(1 - x)^{-1}(\mathbb{1} + x(\mathbb{1} - x)^{-1})^{-1} . \end{aligned}$$

Consider the operator monotone increasing function f_1 from example 2.4.19 and define $z := f_1(a + b)$. Since $a \leq a + b$ we find

$$z \geq f_1(a) = x(\mathbb{1} - x)^{-1}(\mathbb{1} + x(\mathbb{1} - x)^{-1})^{-1} = x$$

and in the same way $z \geq f(b) = y$. It remains to show that $\|z\| < 1$. Since $z \in A_+$ corollary 2.2.2 applies and with corollary 2.3.16 it follows that:

$$\begin{aligned} \|z\| = \rho(z) &= \sup|\sigma(f(a + b))| = \sup|f(\sigma(a + b))| \\ &= \sup|f([0, \|a + b\|])| = \sup\left[0, \frac{1}{1 + \|a + b\|}\right) < 1 . \end{aligned}$$

Next we observe that by corollary 2.1.5 every element $a \in A$ can be written as sum of self adjont elements $a = x + iy$. And by lemma 2.4.5 self adjoint elements can be written as difference of positive elements:

$$a = x + iy = x_+ - x_- + iy_+ - iy_- \ .$$

This means that A is the linear span of A_+ over $\mathbb{C}$. Hence we need only show that $a = \lim_{\Lambda \ni x \to \infty} ax$ for all $a \in A_+$. With corollary 2.3.16 we see[3]that $x(\mathbb{1} - x) \geq 0$ and thus

$$\begin{aligned} 0 \leq x(\mathbb{1} - x) = x - x^2 &= (\mathbb{1} - x) - (\mathbb{1} - 2x + x^2) \\ \Leftrightarrow \qquad (\mathbb{1} - x)^2 &= \mathbb{1} - 2x + x^2 \leq \mathbb{1} - x \ . \end{aligned}$$

For $a \in A_+$ we deduce (corollary 2.4.13):

$$\|a(\mathbb{1} - x)\|^2 = \|(a(\mathbb{1} - x))^* a(\mathbb{1} - x)\| = \|a(\mathbb{1} - x)^2 a\| \leq \|a(\mathbb{1} - x)a\| \ .$$

On the other hand $\alpha f_\alpha(a) \in \Lambda$ for the same reason as $\|z\| < 1$, as long as $\alpha > 0$. Then $(x \overset{!}{=} \alpha f_\alpha(a))$:

$$a(\mathbb{1} - \alpha f_\alpha(a))a = a(\mathbb{1} + \alpha a)^{-1} a \leq \alpha^{-1} a \ .$$

The last step can be seen by using $f(t) = \alpha^{-1} t - t^2(1 + \alpha t) = \frac{1}{\alpha} \frac{t}{1 + \alpha t}$ to show that $0 \leq \alpha^{-1} a - a(\mathbb{1} + \alpha a)a$. By lemma 2.4.15:

$$\|a(\mathbb{1} - \alpha f_\alpha(a))a\| \leq \alpha^{-1} \|a\| \ .$$

Let $\varepsilon > 0$ and $\alpha \geq \varepsilon^{-2}\|a\|$. Choose $x_0 := \alpha f_\alpha(a)$ and let $x \in \Lambda$, such that $x \geq x_0$. Form $x_0 \leq x$ it follows that (using corollary 2.4.13):

$$\begin{aligned} x_0 \leq x \quad &\Leftrightarrow \quad a^* x_0 a = a x_0 a \leq a^* x a = axa \\ &\Leftrightarrow \quad 0 \leq axa - ax_0 a = (a^2 - a_x 0 a) - (a^2 - axa) \\ &\Leftrightarrow \quad a^2 - axa = a(\mathbb{1} - x)a \leq a^2 - ax_0 a = a(\mathbb{1} - x_0)a \\ &\Rightarrow \quad \|a(\mathbb{1} - x)a\| \leq \|a(\mathbb{1} - x_0)a\| \ . \end{aligned}$$

Now we can show that $\|a(\mathbb{1} - x)\| \leq \varepsilon$ for $x \geq x_0$, showing $\|a(\mathbb{1} - x)\| \to 0$ and thus $a(\mathbb{1} - x) \to 0$:

$$\begin{aligned} \|a(\mathbb{1} - x)\| \leq &\|a(\mathbb{1} - x)a\|^{\frac{1}{2}} \leq \|a(\mathbb{1} - x_0)a\|^{\frac{1}{2}} \\ &= \|a(\mathbb{1} - \alpha f_\alpha(a))a\|^{\frac{1}{2}} \leq \alpha^{-\frac{1}{2}} \|a\|^{\frac{1}{2}} \\ &\leq (\varepsilon^{-2}\|a\|)^{-\frac{1}{2}} \|a\|^{\frac{1}{2}} = \varepsilon \ . \end{aligned}$$

□

[3] $f(x) \overset{!}{=} x(\mathbb{1} - x)$ leads to $f(t) = t(1 - t)$. Since $\sigma(x(\mathbb{1} - x)) = f(\sigma(x))$, $t \in \sigma(x) \subset [0, 1)$ it follows that $\sigma(x(1 - x)) \subset \mathbb{R}_{\geq 0}$.

Corollary 2.5.8.
In the proof we have also shown that for $x \in A_+$ with $\|x\| \leq 1$ it holds that

$$x^2 - x = x(x - \mathbb{1}) \geq 0 \ .$$

Proof 2.5.9.
See footnote 3 . ☐

Corollary 2.5.10.
*Let A be a **separable C^*-algebra**, i.e. there is a dense countable subset in A. Then there is a countable approximate unit $(u_k)_{k\in\mathbb{N}}$.*

Proof 2.5.11 ([Mur90, 3.1.1 Remark]).
Let $D = \bigcup_{j\in\mathbb{N}} \{a_j\}$ with $a_j \in A$ denote the dense subset and let $(u_\lambda)_{\lambda\in\Lambda}$ be the approximate unit from theorem 2.5.6. Define $F_n = \{a_1, \dots, a_n\}$, then $f_1 \subset \dots \subset F_n \subset \dots$ and $F = D = \bigcup_{n\in\mathbb{N}} F_n$, while F_n is finite for finite n. Let $\varepsilon > 0$, then there exist $\lambda_1, \dots, \lambda_n \in \Lambda$, such that

$$\|a_j(\mathbb{1} - u_\lambda)\| \leq \varepsilon \qquad \text{for } \lambda \geq \lambda_j \ .$$

Choose $\lambda_\varepsilon \in \Lambda$, such that $\lambda_\varepsilon \geq \lambda_j$ for all $j = 1, \dots, n$. Then $\|a(\mathbb{1} - u_\lambda)\| \leq \varepsilon$ for all $a \in F_n$ and all $\lambda \geq \lambda_\varepsilon$. If n is a positive integer and $\varepsilon = \frac{1}{n}$ then there exists $\lambda_n := \lambda_\varepsilon \in \Lambda$, such that $\|a(\mathbb{1} - u_{\lambda_n})\|$ for all $a \in F_n$. Choosing the λ_n so that $\lambda_n \leq \lambda_{n+1}$ for all n, we obtain a sequence $(u_n)_{n\in\mathbb{N}} := (u_{\lambda_n})_{n\in\mathbb{N}}$, such that

$$\|a(\mathbb{1} - u_n)\| \leq \frac{1}{n} \quad \forall\, a \in F_n \ .$$

Thus $\lim_{n\to\infty} \|a(\mathbb{1} - u_n)\| = 0$ for all $a \in F$. This construction does not depend on the order of elements in D, such that reordering is allowed. Let $a \in A$ be arbitrary and $(a_n)_{n\in\mathbb{N}}$ be a sequence in F, such that $a = \lim_{n\to\infty} a_n$. This is possible since F is dense in A. Then by the previous construction:

$$\|a_n(\mathbb{1} - u_n)\| \leq \frac{1}{n}$$

$$\Rightarrow \quad \|a_n(\mathbb{1} - u_n)\| \to 0 = \|a(\mathbb{1} - u_\infty)\| = \lim_{n\to\infty} \|a(\mathbb{1} - u_n)\| \ .$$

☐

Lemma 2.5.12.
Let $x^*x \leq a$. For all $0 < \alpha < \frac{1}{2}$ there is a $b \in A$ such that $\|b\| \leq \|a^{\frac{1}{2}-\alpha}\|$ and $x = ba^\alpha$.

Proof 2.5.13.
Define

$$g_n(t) = \frac{t^{1-\alpha}}{(\frac{1}{n}+t)^{\frac{1}{2}}} \ , \qquad d_m n = \left(\frac{1}{m}+a\right)^{-\frac{1}{2}} - \left(\frac{1}{n}+a\right)^{-\frac{1}{2}} \ .$$

The sequence (g_n) is increasing an converges against $g(t) = t^{\frac{1}{2}-\alpha} = \mathbb{1}^{\frac{1}{2}-\alpha}(t)$. By Dini's theorem it converges uniformly on the compact space $[0, \|a\|]$. Consider the sequence $b_n = x(\frac{1}{2}+a)^{-\frac{1}{2}}a^{\frac{1}{2}-\alpha}$. With corollary 2.3.16, corollary 2.4.13 and since $0 \leq x^*x \leq a$ by theorem 2.4.9 as well as $d_{mn} \in C^*(a, \mathbb{1})$ we find

$$\begin{aligned} \|b_m - b_n\|^2 = \|xd_m n a^{\frac{1}{2}-\alpha} &= \|a^{\frac{1}{2}-\alpha} d_{mn} x^* x d_{mn} a^{\frac{1}{2}-\alpha}\| \\ &\leq \|a^{\frac{1}{2}-\alpha} d_{mn} a d_{mn} a^{\frac{1}{2}-\alpha}\| \\ &= \|d_{mn} a^{1-\alpha}\|^2 = \|g_m(a) - g_n(a)\|^2 \\ &\leq \|g_m - g_n\|_\infty^2 \longrightarrow 0 \ , \end{aligned}$$

that is, (b_n) is a Cauchy sequence and thus convergent. Let $b = \lim\limits_{n\to\infty} b_n$, then

$$\|b\| \leq \sup_n \|b_n\| = \sup_n \|a^{\frac{1}{2}-\alpha}(\tfrac{1}{n}+a)^{-\frac{1}{2}} x^* x (\tfrac{1}{n}+a)^{-\frac{1}{2}} a^{\frac{1}{2}-\alpha}\| \leq \|a^{\frac{1}{2}-\alpha}\|$$

and also:

$$ba^\alpha = \lim_{n\to\infty} x(\tfrac{1}{n}+a)^{-\frac{1}{2}} a^{\frac{1}{2}} = x \ .$$

□

For the next corollary we define the absolute value on a C^*-algebra.

Definition 2.5.14.
Let A be a C^*-algebra and $a \in A$, define the **absolute value** $|a| = (a^*a)^{\frac{1}{2}}$.

With this definition we find:

Corollary 2.5.15.
Let $a \in A$ and $0 < \alpha < 1$, then there is a $u \in A$, such that $a = u|a|^\alpha$.

Proof 2.5.16.
By definition it holds that $a^*a = |a|^2$. By lemma 2.5.12 there is a $u \in A$ such that $(\beta = \frac{\alpha}{2}$, such that $0 < \beta < \frac{1}{2})$:

$$a = u(a^*a)^\beta = u(a^*a)^{\frac{\alpha}{2}} = u(|a|^2)^{\frac{\alpha}{2}} = u|a|^\alpha \ .$$

□

2.6 Ideals and quotients

Definition 2.6.1.
A cone $C \subset A_+$ is called **hereditary**, if from $0 \leq a \leq b$ and $b \in C$ it follows that $a \in C$.

We have met the concept of convex cones already. A **cone** is less restrictive, demanding that the subset $C \subset V$ of a vector space V satisfies

$$\alpha v \in C \qquad \forall\ \alpha \in \mathbb{R}_{>0}\ v \in C\ .$$

Lemma 2.6.2.
Let $L \subset A$ be a closed left ideal and define $L_+ := L \cap A_+$. Then L_+ is hereditary and for all $a \in A$ it holds that $a \in L$ if and only if $a^*a \in L_+$.

Proof 2.6.3.
Let $x \in A$ and $b \in L_+$, such that $x^*x \leq b$. By lemma 2.5.12 there is a $c \in A$ such that $x = cb^\alpha$. Since $b \in L \cap A_+$ it holds that $b^* = b$, thus[4] $C^*(b) \subset L$ and also $b^\alpha \in L$ by lemma 2.3.12. Then since L is a left ideal, $x = cb^\alpha \in L$.

First we show now that L is a hereditary cone. That L is a cone follows from the property to be a left ideal, since $\alpha a = (\alpha e)a \in L$ for $a \in L$. Assume now that $0 \leq a \leq b$ and $b \in L$. By theorem 2.4.9 there is a $x \in A$, such that $a = x^*x \leq b$ and by the previous reasoning $a \in L$, showing that L is hereditary.

The second claim has an easy direction. If $a \in L$, then $a^*a \in L$, since L is a left ideal. But by theorem 2.4.9 $a^*a \in A_+$, hence $a^*a \in L_+$. For the opposite direction let $a^*a \in L_+$. By theorem 2.4.9 $a^*a + y \geq 0$ for $y \in L_+$ and thus $b := a^*a + y \in L_+$ with $a^*a \leq b$. By the first paragraph it follows that $a \in L$. □

Corollary 2.6.4.
Let $J \subset A$ be a closed ideal, then it holds that $J^* = J$.

Proof 2.6.5.
Positive elements are self adjoint by lemma 2.4.2. Hence $(J^*)_+ = J_+ = (J_+)^*$. From lemma 2.6.2 it follows that

$$a \in J \quad \Rightarrow \quad a^*a \in J_+ = (J^*)_+ \quad \Rightarrow a \in J^* \quad \Rightarrow \quad a^* \in J\ .$$

□

This corollary also shows that J is a C^*-sub algebra. For the next lemma we recall the meaning of the quotient norm (see footnote 1 on page 17).

[4] For $C^*(b)$ to be a C^*-algebra, the closure of the set of polynomials is taken, such that L needs to be closed, so as to contain $C^*(b)$.

Lemma 2.6.6.
Let $J \subset A$ be a closed ideal and $(u_\lambda)_{\lambda\in\Lambda}$ its approximate unit, then for all $a \in A$ it holds that

$$\|a\|_{A/J} = \lim_{\lambda\to\infty} \|a - au_\lambda\| \ .$$

Proof 2.6.7.
Let $\varepsilon > 0$ and $b \in J$ such that $\|a - b\| \le \|a\|_{A/J} + \frac{\varepsilon}{2}$. Since $\mathbb{1} - u_\lambda \le \mathbb{1}$ it follows that $\|\mathbb{1} - u_\lambda\| \le \|\mathbb{1}\| = 1$ by corollary 2.4.13 and

$$\begin{aligned}\|a - au_\lambda\| &= \|a - au_\lambda - b(\mathbb{1} - u_\lambda) + b(\mathbb{1} - u_\lambda)\| \\ &= \|(a - b)(\mathbb{1} - u_\lambda) + b(\mathbb{1} - u_\lambda)\| \\ &\le \|(a - b)(\mathbb{1} - u_\lambda)\| + \|b(\mathbb{1} - u_\lambda)\| \\ &\le \|a - b\|\|\mathbb{1} - u_\lambda\| + \|b(\mathbb{1} - u_\lambda)\| \\ &\le \|a - b\| + \|b - bu_\lambda\| \le \|a\|_{A/J} + \frac{\varepsilon}{2} + \|b - bu_\lambda\| \ .\end{aligned}$$

Chose $\lambda_0 \in \Lambda$, such that $\|b - bu_\lambda\| \le \frac{\varepsilon}{2}$ for all $\lambda \ge \lambda_0$, then

$$\|a\|_{A/J} = \inf\{\|a - b\| \mid b \in J\} \le \|a - au_\lambda\| \le \|a\|_{A/J} + \varepsilon \ .$$

□

Theorem 2.6.8 (**I. Segal**)**.**
Let $J \subset A$ be a closed ideal, then A/J with the quotient norm is a C^*-algebra.

Proof 2.6.9.
By lemma 1.4.24, the quotient A/J is a Banach algebra. From corollary 2.6.4 we know that $J^* = J$, such that $*$ from A induces a well defined $*$ on A/J. It remains to show the C^*-property of the quotient norm.

Let $a \in A$, then by lemma 2.6.6:

$$\|a\|^2_{A/J} = \lim_{\lambda\to\infty} \|a(\mathbb{1} - u_\lambda)\|^2 = \lim_{\lambda\to\infty} \|(\mathbb{1} - u_\lambda)a^*a(\mathbb{1} - u_\lambda)\| \le \|a^*a\| \ ,$$

since $\|\mathbb{1} - u_\lambda\| \le 1$, as was seen in the proof of lemma 2.6.6. □

Theorem 2.6.10.
Let $\phi\colon A \to B$ be a $*$-morphism between two C^*-algebras, then ϕ is norm decreasing:

$$\|\phi(a)\|_B \le \|a\|_A \ .$$

If ϕ is also injective, then it also is an isometry.

Proof 2.6.11.
By corollary 2.3.18 it holds that $\sigma_B(\phi(a)) \subset \sigma_A(a)$. Assume now that $a \in A$ is self adjoint, then with corollary 2.2.2 it follows that:

$$\|\phi(a)\| = \rho_B(\phi(a)) \leq \rho_A(a) = \|a\| \ .$$

For an arbitrary $a \in A$ it follows that:

$$\|\phi(a)\| = \|\phi(a^*a)\|^{\frac{1}{2}} \leq \|a^*a\|^{\frac{1}{2}} = \|a\| \ .$$

From this equation together with theorem 2.4.9 it is enough to show isometry for positive elements. Let ϕ be injective and assume it to be no isometry on positive elements. Then there is an $a \geq 0$, such that:

$$r := \|\phi(a)\| < \|a\| =: s \ .$$

Let $f \in C([0,s])$ with $f([0,r]) = 0$ and $f(s) = 1$. By lemma 2.3.18 it follows that

$$0 = \phi(f(a)) = f(\phi(a)) \ .$$

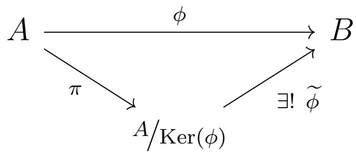
Yet since ϕ is injective and $a \neq 0$ it holds that $\phi(a) \neq 0$. But since the functional calculus is defined by an isomorphsism, $f(b) \neq 0$ for all $b \neq 0$, hence we have a contradiction. Thus the assumption $\|\phi(a)\| < \|a\|$ was wrong.

It remains to show that $\phi(A)$ is a C^*-algebra. For this, we will follow [Mur90, proof of theoeem 3.1.6]. The induced map ${}^{A}/_{\mathrm{Ker}(\phi)} \to B$ is an injective $*$-morphism and by the previous part of the proof an isometry, hence mapping complete spaces to a complete image. Hence $\phi(A)$ is a C^*-algebra. □

Corollary 2.6.12.
Let $\phi\colon A \to B$ be a $*$-morphism, then ϕ is closed in the norm topology.

Proof 2.6.13.
First we consider the case of an injective $\phi\colon A \to B$. From theorem 2.6.10 it hollows that ϕ is an isometry. So it especially maps Cauchy-sequences onto Cauchy-sequences. Hence it maps complete spaces to a complete image. But then the limit a of the sequence (a_n) is mapped to the limit $b = \phi(a)$ of the sequence $(b_n) = (\phi(a_n))$. Since $\phi(A)$ is complete, it follows that $b \in \phi(A)$. Thus, ϕ maps closed sets onto closed sets. (In metric spaces, the closure contains all limit point of sequences, i.e. the special cases of nets). This shows that injective $*$-morphism are closed.

Now consider the case, where ϕ is not necessarily injectve. Then, because of the fundamental theorem on homomorphisms/the universal property, $\exists!\ \widetilde{\phi}\colon {}^{A}/_{\mathrm{Ker}(\phi)} \to B$, such that the following diagram commutes:



However, $\widetilde{\phi}$ is injective and thus closed. Since ϕ and $\widetilde{\phi}$ have the same image, because of the commutativity of the diagram, it follows that ϕ is closed. □

Corollary 2.6.14.
Every $*$-morphism between C^*-algebras is continuous in the norm topology.

Proof 2.6.15.
Let $\phi\colon A \to B$ be a $*$-morphism, then by theorem 2.6.10 it holds that $\|\phi(a)\|_B \leq \|a\|_A$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$, then

$$\|a - b\| \leq \varepsilon \quad \Rightarrow \quad \|\Phi(a) - \phi(b)\| = \|\phi(a - b)\| \leq \|a - b\| \leq \varepsilon \ ,$$

for all $a, b \in A$. □

Corollary 2.6.16.
Let $J \subset A$ be a closed ideal and $B \subset A$ a C^*-sub algebra, then $B + J$ is a C^*-sub algebra and

$$^{(B+J)}\!/_{J} \cong {}^{B}\!/_{B\cap J} \ .$$

Proof 2.6.17.
Let $\pi\colon A \to {}^{A}\!/_{J}$ be the canonical projection, i.e. a $*$-morphism. Then by theorem 2.6.10 is complete and thus closed and so is $B + J = \pi^{-1}(\pi(B))$. Thus $B + J$ is a C^*-sub algebra.

Consider the restriction $\phi = \pi|_{B+J}\colon B + J \to {}^{(B+J)}\!/_{J}$. This map is a ringisomorphism and since ϕ is also a $*$-morphism the isomorphy holds in the $C*$-algebra sense. □

2.7 Positive linear forms

Definition 2.7.1.
Let $\omega\colon A \to \mathbb{C}$ be a linear functional. It is called **positive**, if $\omega(A_+) \subset \mathbb{R}_{\geq 0}$. It is called **state** if it satisfies furthermore:

$$\|\omega\| := \{|\omega(a)| \mid \|a\| \leq 1\} = 1 \ .$$

If ω is a positive linear functional, then it follows that for all $a \in A$:

$$\omega(a^*) = \omega((x_+ + iy_+ - x_- - iy_-)^*) = \omega(x_+ - iy_+ - x_- + iy_-)$$

$$
\begin{aligned}
&= \omega(x_+) - i\omega(y_+) - \omega(x_-) + i\omega(y_-) \\
&= \overline{\omega(x_+)} + \overline{i\omega(y_+)} - \overline{\omega(x_-)} - \overline{i\omega(y_-)} \\
&= \overline{\omega(x_+ + iy_+ - x_- - iy_-)} = \overline{\omega(a)} = \overline{\omega}(a) \ ,
\end{aligned}
$$

since $x_\pm$ and $y_\pm$ are positive and $\omega(x_\pm)$ as well as $\omega(y_\pm)$ are real.

Corollary 2.7.2.
Let ϕ be a positive linear functional and $a \leq b$, then $\phi(a) \leq \phi(b)$.

Proof 2.7.3.
By definition $b - a \geq 0$. Since ϕ is positive it holds that $\phi(b-a) \geq 0$, such that:

$$
0 \leq \phi(b-a) = \phi(b) - \phi(a) \qquad \Leftrightarrow \phi(a) \leq \phi(b) \ .
$$

□

Theorem 2.7.4.
Let ϕ be a positive linear functional, then the Cauchy-Schwarz inequality

$$
|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)
$$

is satisfied for all $a, b \in A$.

Proof 2.7.5.
Since a^*a and b^*b are positive, Œ we can assume that $\phi(b^*a) \neq 0$. Define $z := -t\phi(a^*b)|\phi(b^*a)|^{-1}$ for $t > 0$. Then

$$
\begin{aligned}
0 &\leq \phi\left((za+b)^*(za+b)\right) = |z|^2\phi(a^*a) + \overline{z}\phi(a^*b) + z\phi(b^*a) + \phi(b^*b) \\
&= t^2\phi(a^*a) - 2t|\phi(b^*a)| + \phi(b^*b) \ ,
\end{aligned}
$$

$$
\Leftrightarrow \qquad |\phi(b^*a)| \leq \frac{1}{2}(t\phi(a^*a) + \tfrac{1}{t}\phi(b^*b) \ .
$$

Let now (a_n) and (b_n) be sequences in $\mathbb{R}_{>0}$ such that $a_n \to \sqrt{\phi(a^*a)}$ and $b_n \to \sqrt{\phi(b^*b)}$. Chose $t_n = \frac{b_n}{a_n}$, then

$$
|\phi(b^*a)| \leq \lim_{n\to\infty} \frac{1}{2}(t_n\phi(a^*a) + \tfrac{1}{t_n}\phi(b^*b) = \sqrt{\phi(a^*a)\phi(b^*b)} \ ,
$$

which is the desired result after squaring both sides. □

Lemma 2.7.6.
Positive linear functionals are continuous.

Proof 2.7.7.
Let ϕ be positive and assume that ϕ is not bounded on

$$B_+ := \{a \in A_+ \mid \|a\| \leq 1\} \ .$$

Then there is an $a_k \in B_+$ such that $\phi(a_k) \geq 2^{k+1}$, for all $k \in \mathbb{N}$. Then because of theorem 2.4.9:

$$a := \sum_{k=0}^{\infty} 2^{-k+1} a_k \in B_+ \ .$$

For all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \phi(a) &\geq \phi\left(\sum_{k=0}^{0} 2^{-(k+1)} a_k\right) = \sum_{k=0}^{n} 2^{-(k+1)} \phi(a_k) \\ &\geq \sum_{k=0}^{n} 2^{-(k+1)} 2^{k+1} = \sum_{k=0}^{n} 1 = n+1 \ , \end{aligned}$$

which is a contradiction to $\phi(a) \in \mathbb{R}$. This shows that ϕ is bounded on B_+. Let $a \in A$ with $\|a\| \leq 1$, then the standard decomposition $a = x + iy = x_+ + ix_+ + y_- - iy_-$

$$\|x_\pm\| \leq \|x\| \leq \|a\| \leq 1 \qquad \text{and} \qquad \|y_\pm\| \leq \|y\| \leq \|a\| \leq 1 \ ,$$

as can be seen by using the explicit formula from corollary 2.1.5 for $\|y\| \leq \|a\|$ as well as $\|x\| \leq \|a\|$ and using corollary 2.4.13 for $\|x_\pm\| \leq \|x\|$ as well as $\|y_\pm\| \leq \|y\|$. Then

$$|\phi(a)| \leq |\phi(x_+)| + |\phi(x_-)| + |\phi(y_+)| + |\phi(y_-)| \leq 4\|\phi\| < \infty \ .$$

Hence ϕ is bounded on the unit ball of A and thus continuous. □

Lemma 2.7.8.
Let $a, b \in A_+$ such that $\|a\| \leq 1$ and $\|b\| \leq 1$, then it holds that

$$\|a - b\| \leq 1 \ .$$

Proof 2.7.9.
From $a \geq 0$ as well as $\|a\| \leq 1$ and lemma 1.4.7 it follows that $\sigma(a) \subset [0,1]$. In $C([0,1])$ it holds that $0 \leq \mathrm{Id} \leq 1$, such that $0 \leq a \leq \mathbb{1}$. The same holds true for b. Then

$$a \leq \mathbb{1} \quad \Leftrightarrow \quad \mathbb{1} - a = \mathbb{1} - b - (a - b) \geq 0 \quad \overset{b \leq \mathbb{1}}{\Rightarrow} \quad a - b \leq \mathbb{1} - b \leq \mathbb{1} \ .$$

In the same way one sees that $b - a \leq 1$.Œ $a - b \geq 0$, otherwise choose $b - a$. With corollary 2.4.13 it follows that

$$\|a - b\| = \|\mathbb{1}\| = 1 \ .$$

□

Theorem 2.7.10.
A linear functional ϕ on A is positive if and only if

$$\infty > \|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) \ ,$$

for one (then for all) approximate unit $(u_\lambda)_{\lambda\in\Lambda}$. If A is unital and ϕ is continuous, then this condition is equivalent to $\phi(\mathbb{1}) = \|\phi\|$.

Proof 2.7.11.

"⇒"

Let ϕ be positive and $(u_\lambda)_{\lambda\in\Lambda}$ an approximate unit. As approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ is an increasing net, i.e. $u_\lambda \geq u_\mu$ for $\lambda \geq \mu$. Then from $\phi \geq 0$ it follows that

$$\begin{aligned} y \geq x \quad \Leftrightarrow \quad y - x \geq 0 \quad &\Rightarrow \quad 0 \leq \phi(y-x) = \phi(y) - \phi(x) \\ &\Leftrightarrow \quad \phi(y) \geq \phi(x) \ . \end{aligned}$$

Hence $(\phi(u_\lambda))_{\lambda\in\Lambda}$ is an increasing net in $\mathbb{C}$ with limit $\alpha \leq \|\phi\|$. For $\|a\| \leq 1$ it holds that:

$$\begin{aligned} |\phi(u_\lambda a)|^2 &\leq \phi(u_\lambda^2)\phi(a^*a) \leq \phi(u_\lambda)\|\phi\|\|a^*a\| \\ &\leq \alpha\|\phi\|\|a\|^2 \leq \alpha\|\phi\| \ , \end{aligned}$$

using theorem 2.7.4 and corollary 2.5.8 showing that $u_\lambda^2 \leq u_\lambda$. Taking the limit we find that $|\phi(a)|^2 \leq \alpha\|\phi\|$ for all a with $\|a\| \leq 1$. By the definition of $\|\phi\|$ this yields (using that already $\alpha \leq \|\phi\|$):

$$\begin{aligned} \|\phi\|^2 \leq \alpha\|\phi\| \quad &\Leftrightarrow \quad \|\phi\| \leq \alpha \\ \Rightarrow \quad &\|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) < \infty \ . \end{aligned}$$

The continuity of ϕ follows from lemma 2.7.6.

"⇐"

For the opposite direction, $\|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda)$ for an approximate unit $(u_\lambda)_{\lambda\in\lambda} < \infty$. Since $\|\phi\| < \infty$ it follows that ϕ is continuous, as it is the operator norm. First we show that ϕ is real on A_{sa}. Let $a \in A_{sa}$ with $\|a\| \leq 1$ and write $\phi(a) = x + iy$ for $x, y \in \mathbb{R}$. Œ $y \geq 0$. Let $n \geq 1$ arbitrary and $\lambda_0 \in \Lambda$, such that for all $\lambda \geq \lambda_0$

$$\|au_\lambda - u_\lambda a\| \leq \frac{1}{n}$$

holds. Then for all $\lambda \geq \lambda_0$:

$$\begin{aligned} \|nu_\lambda - ia\|^2 &= \|n^2u_\lambda^2 + a^2 - in(au_\lambda - u_\lambda a)\| \\ &\leq n^2\|u_\lambda\|^2 + \|a\|^2 + n\|au_\lambda - u_\lambda a\| \\ &\leq n^2 + 2 \ . \end{aligned}$$

It follows that

$$\begin{aligned}(n\|\phi\|+y)^2+x^2 = |n\|\phi\|+y-ix|^2 &= \lim_{\lambda\in\Lambda}|\phi(nu_\lambda - ia)|^2 \\ &\le \lim_{\lambda\in\Lambda}\|\phi\|^2\|nu_\lambda - ia\|^2 \le (n^2+2)\|\phi\|^2 \ .\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad n^2\|\phi\|^2+2\|\phi^2\| \ge & n\|\phi\|+y^2+x^2 \\ &= n^2\|\phi\|^2+x^2+y^2+2ny\|\phi\| \\ &= n^2\|\phi\|^2+|\phi(a)|^2+2ny\|\phi\|\end{aligned}$$

$$\Leftrightarrow\qquad 2\|\phi\|^2 \ge |\phi(a)|^2+2ny\|\phi\| \ .$$

Since n was chosen arbitrarily, it is necessary that $y=0$ and thus $\phi(a)\in\mathbb{R}$. Let now $a\in A_+$ with $\|a\|\le 1$. Because of lemma 2.7.8 it holds that $\|u_\lambda - a\|\le 1$ and thus

$$\begin{aligned}\|\phi\|-\phi(a) = \lim_{\lambda\in\Lambda}\phi(u_\lambda - a) &\le \lim_{\lambda\in\Lambda}|\phi(u_\lambda - a)| \\ &\le \lim_{\lambda\in\Lambda}\|\phi\|\|u_\lambda - a\| \le \|\phi\| \ .\end{aligned}$$

Hence $\phi(a)\ge 0$. The rest is a matter of scaling, such that $\phi\ge 0$.

unital A

Finally, let A be unital and ϕ be continuous. Then $\mathbb{1}=\lim_{\lambda\in\Lambda}u_\lambda$. By the continuity of ϕ it follows that:

$$\|\phi\| = \lim_{\lambda\in\Lambda}\phi(u_\lambda) = \phi(\lim_{\lambda\in\Lambda}u_\lambda) = \phi(\mathbb{1}) \ .$$

□

2.8 Representations and the GNS-construction

Definition 2.8.1.
Let $\mathcal{H}$ be a Hilbert space. A **$*$-representation** of A is a $*$-morphism $\pi\colon A\to\mathcal{L}(\mathcal{H})$. A vector $\psi\in\mathcal{H}$ is called **cyclical**, if $\pi(A)\psi$ is dense in $\mathcal{H}$. The representation is called **non-degenerate**, if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.
A $*$-representation is called **faithful**, if it is injective.

A property of nets is, that for a subset $U\subset X$ of a topological space the point $x\in\overline{U}$ if and only if there is a net (x_α) such that $\lim_\alpha x_\alpha = x$.

Corollary 2.8.2.
Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit. A representation π is non-degenerate, if and

only if

$$\psi = \lim_{\lambda\in\Lambda} \pi(u_\lambda)\psi \qquad \forall\ \psi \in \mathcal{H}\ .$$

Proof 2.8.3.
Assume that $\psi = \lim_{\lambda\in\Lambda} \pi(u_\lambda)\psi$ holds for all $\psi \in \mathcal{H}$. Then, since $\pi(u_\lambda)$ is a net in $\mathcal{H}$, ψ is in the closure of $\pi(A)\mathcal{H}$.

For the opposite direction, assume π to be non-degenerate. Let now $\psi \neq 0 \in \mathcal{H}$ and $\varepsilon > 0$. Since $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$, there are $a \in A$ and $\xi \neq 0 \in \mathcal{H}$, such that $\|\psi - \pi(a)\xi\| \leq \frac{\varepsilon}{3}$. Furthermore, there is a λ_0 such that $\|a - u_\lambda a\| \leq \frac{\varepsilon}{3\|\xi\|}$ for all $\lambda \geq \lambda_0$. Then it follows that (using theorem 2.6.10)

$$\begin{aligned}
\|\psi - \pi(u_\lambda)\psi\| &= \|\psi - \pi(a)\xi + \pi(a)\xi - \pi(u_\lambda)\pi(a)\xi + \pi(u_\lambda)\pi(a)\xi - \pi(u_\lambda)\psi\| \\
&\leq \|\psi - \pi(a)\xi\| + \|\pi(a)\xi - \pi(u_\lambda)\pi(a)\xi\| + \|\pi(u_\lambda)\pi(a)\xi - \pi(u_\lambda)\psi\| \\
&\leq \frac{\varepsilon}{3} + \|\pi(a - u_\lambda a)\| \cdot \|\xi\| + \|\pi(u_\lambda)\|\ \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \|a - u_\lambda a\|\|\xi\| + \|u_\lambda\|\frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3\|\xi\|}\|\xi\| + \frac{\varepsilon}{3} = \varepsilon\ .
\end{aligned}$$

□

Example 2.8.4.
Let $(\mathcal{H}, \pi)$ be a $*$-representation of A and $\psi \in \mathcal{H}$. Then ϕ, defined by

$$\phi(a) := \langle\psi|\pi(a)\psi\rangle \qquad \forall\ a \in A\ ,$$

is a positive linear functional. Let π be non-degenerate, then ϕ is a state if and only if $\|\psi\| = 1$.

Proof 2.8.5.
By theorem 2.4.9 every positive $a \in A_+$ can be written as $a = b^*b$. Then it follows that:

$$\begin{aligned}
\phi(b^*b) &= \langle\psi|\pi(b^*b)\psi\rangle = \langle\psi|\pi(b)^\dagger\pi(b)\psi\rangle \\
&= \langle\pi(b)\psi|\pi(b)\psi\rangle = \|\pi(b)\psi\|^2 \geq 0\ .
\end{aligned}$$

Let now π be non-degenerate, then by corollary 2.8.2 it holds that $\psi = \lim_{\lambda\in\Lambda} \pi(u_\lambda)\psi$. By theorem 2.7.10:

$$\begin{aligned}
\|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) = \lim_{\lambda\in\Lambda} \langle\psi|\pi(u_\lambda)\psi\rangle = \\
\langle\psi|\lim_{\lambda\in\Lambda} \pi(u_\lambda)\psi\rangle = \langle\psi|\psi\rangle = \|\psi\|^2\ .
\end{aligned}$$

Hence $\|\phi\| = 1$ if and only if $\|\psi\| = 0$, which is the claim. □

In the following we will construct a non-degenerate representation. This construction is called **GNS construction**, named after Gelfand, Naimark and Segal. Let ϕ be a positive linear functional on A and define

$$N := \{a \in A \mid \phi(a^*a) = 0\} \ .$$

Corollary 2.8.6.
N is a sub vector space of A and $\langle [b] \mid [a] \rangle_\phi := \phi(b^*a)$ is a well defined hermitian scalar product on the quotient A/N.

Proof 2.8.7.
Because of the Cauchy-Schwarz inequality (theorem 2.7.4) it holds that $\phi(b^*a) = 0$ if $a \in N$ or $b \in N$. Thus

$$\phi((a+b)^*(a+b)) = \phi(a^*a) + \phi(a^*b) + \phi(b^*b) + \phi(b^*a) = 0 \qquad \forall\ a, b \in A \ ,$$

showing that N is indeed a sub vector space. Furthermore the Cauchy-Schwarz inequality shows that $\langle \cdot \mid \cdot \rangle_\phi$ is well defined. Let $a', b' \in N$, then:

$$\begin{aligned}\phi((b+b')^*(a+a')) &= \phi(b^*a) + \phi(b^*a') + \phi(b'^*a) + \phi(b'^*a') \\ &= \phi(b^*a) = \langle [b] \mid [a] \rangle_\phi \ .\end{aligned}$$

By construction $\langle \cdot \mid \cdot \rangle_\phi$ is hermitian sesquilinear and $\langle [a], [a] \rangle_\phi = \phi(a^*a) \geq 0$ since ϕ is positive. It remains to show definiteness. If $[a] = [0]$, then $\langle [0] \mid [0] \rangle_\phi = \phi(0^*0) = 0$. On the other hand, if $\langle [a] \mid [a] \rangle_\phi = 0$, then $\phi(a^*a) = 0$ and thus $a \in N$, which means that $[a] = [0]$. □

The space $(A/N, \langle \cdot \mid \cdot \rangle_\phi)$ is a pre Hilbert space, which is not yet complete. Let $\mathcal{H}_\phi$ denote the completion with respect to the norm induced by $\langle \cdot \mid \cdot \rangle_\phi$ and denote the extended inner product by $\langle \cdot \mid \cdot \rangle$.

Let $b \in N$ and $a \in A$, then it follows that

$$\phi((ab)^*ab) = \phi((a^*ab)^*b) = 0$$

it follows that $ab \in N$, which shows that N is a left ideal of A. Furthermore, A acts by left multiplication on A/N:

$$a \triangleright (b + N) := ab + N \ .$$

Theorem 2.8.8.
Let ϕ be a positive linear functional. The action of A on A/N extends uniquely to a non-degenerate $*$-representation π_ϕ on $\mathcal{H}_\phi$. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit, then $[u_\lambda]$ converges against a cyclical vector $\psi_\phi \in \mathcal{H}_\phi$. Furthermore it holds that

$$\phi(a) = \langle \psi_\phi \mid \pi_\phi(a)\psi_\phi \rangle \qquad \forall\ a \in A \ .$$

Proof 2.8.9.
Let $a, b \in A$. Because of corollary 2.4.13, since $a^*a \in A_+$ it holds that $a^*a \leq \|a^*a\| = \|a\|^2$ and thus $b^*a^*ab \leq \|a\|^2 b^*b$. Then with corollary 2.7.2:

$$\|a \triangleright [b]\|_\phi^2 = \phi((ab)^*ab) = \phi(b^*a^*ab) \leq \|a\|^2 \phi(b^*b) = \|a\|^2 \|[b]\|_\phi^2 \ .$$

Thus the operator $a\triangleright$ is bounded and linear. By the bounded linear transformation theorem, it extends uniquely to a linear bounded operator $\pi_\phi(a)$ on the completion $\mathcal{H}_\phi$. By construction, π_ϕ is also multiplicative. Because of

$$\langle [b] \mid \pi_\phi(a)[c] \rangle = \phi(b^*ac) = \phi((a^*b)^*c) = \langle \pi_\phi(a^*)[b] \mid [c] \rangle \qquad \forall\ a, b, c \in A$$

$$\Rightarrow \qquad \pi(a^*) = \pi(a)^\dagger$$

π_ϕ is a $*$-morphism into $\mathcal{L}(\mathcal{H}_\phi)$. By construction of π_ϕ it holds that ${}^{A}\!/_{N} \subset \pi_\phi(A)\mathcal{H}_\phi$, which shows that π_ϕ is non-degenerate, since ${}^{A}\!/_{N}$ is dense in $\mathcal{H}_\phi$ as metric completion.

Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of A. Because of corollary 2.5.8 and lemma 2.7.8 it follows that $u_\mu - u_\lambda \geq (u_\mu - u_\lambda)^2$ for all $\lambda \leq \mu$. Thus with corollary 2.7.2 and theorem 2.7.10 it follows that:

$$\begin{aligned} \|[u_\mu] - [u_\lambda]\|^2 =& \phi((u_\mu - u_\lambda)^*(u_\mu - u_\lambda) = \phi((u_\mu - u_\lambda)^2) \\ &\leq \phi(u_\mu - u_\lambda)^2 \to \|\phi\| - \|\phi\| = 0 \ . \end{aligned}$$

This means, that $([u_\lambda])_{\lambda \in \Lambda}$ is a Cauchy net in $\mathcal{H}_\phi$ and thus converges against a $\psi_\phi \in \mathcal{H}_\phi$. For all $a \in A$ it follows that

$$\pi_\phi(a)\psi_\phi = \lim_{\lambda \in \Lambda} [au_\lambda] = [\lim_{\lambda \in \Lambda} au_\lambda] = [a]$$

This shows that $\pi_\phi(A)\psi_\phi = {}^{A}\!/_{N}$, i.e. $\pi_\phi(A)\psi_\phi$ is dense in $\mathcal{H}_\phi$, proving that ψ_ϕ is cyclic.

As seen in the proof of theorem 2.5.6, A is the linear span of A_+, such that we only need to show the equation for positive elements, because of its linearity. Since all positive elements can be written as a^*a we have:

$$\begin{aligned} \langle \psi_\phi \mid \pi_\phi(a^*a)\psi_\phi \rangle &= \|\pi_\phi(a)\psi_\phi\|_\phi^2 = \langle \pi_\phi(a)\psi_\phi \mid \pi_\phi(a)\psi_\phi \rangle \\ &= \langle [a] \mid [a] \rangle = \phi(a^*a) \ . \end{aligned}$$

□

Corollary 2.8.10.
Let ϕ be a positive linear functional on A and extend it to $\widetilde{\phi}\colon \widetilde{A} \to \mathbb{C}$ by $\widetilde{\phi}(\mathbb{1}) := \|\phi\|$. Then $\widetilde{\phi}$ is the unique linear extension of ϕ to a positive linear functional on $\widetilde{A}$. Furthermore it holds that $\|\widetilde{\phi}\| = \|\phi\|$.

Proof 2.8.11.
By the Hahn-Banach theorem, there exists an extension $\phi'\colon \widetilde{A} \to \mathbb{C}$, such that $\phi'|_A \equiv \phi$ and $\|\phi'\| = \|\phi\|$. Uniqueness follows from theorem 2.7.10, because $\widetilde{A}$ is unital and thus:

$$\phi'(\mathbb{1}) = \|\phi'\| = \|\phi\| = \widetilde{\phi}(\mathbb{1}) \ .$$

Since every $\tilde{a} \in \widetilde{A}$ can be written as $a + z\mathbb{1}$, this proves uniqueness.

Let $(\mathcal{H}, \pi)$ be the GNS representation (theorem 2.8.8) w.r.t. ϕ and let $\psi \equiv \psi_\phi$. Then:

$$\phi(a) = \langle \psi \mid \pi(a)\psi \rangle \qquad \forall\ a \in A \ .$$

Furthermore, by construction, it holds that $\pi(a)\psi = [a]$, and for every approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ it holds that $[u_\lambda] \to \psi$. Since $\pi(a)$ is bounded on A/N, it is continuous, allowing to exchange limits, such that:

$$\begin{aligned}\lim_\lambda \pi(u_\lambda)\psi &= \lim_\lambda \pi(u_\lambda) \lim_\mu [u_\mu] = \lim_{\lambda,\mu} \pi(u_\lambda)[u_\mu] = \lim_{\lambda,\mu}[u_\lambda u_\mu] \\ &= \lim_\mu [\lim_\lambda u_\lambda u_\mu] = \lim_\mu [u_\mu] = \psi \ .\end{aligned}$$

Also by the construction, an extension of π to a representation of $\widetilde{A}$ is given by $\pi(\mathbb{1})[a] = [\mathbb{1}a] = [a]$ such that $\pi(\mathbb{1}) = \mathrm{Id}_{\mathcal{H}}$. With theorem 2.7.10 it follows that:

$$\widetilde{\phi}(\mathbb{1}) = \|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) = \lim_{\lambda\in\Lambda} \langle \psi \mid \pi(u_\lambda)\psi \rangle = \langle \psi \mid \psi \rangle = \|\psi\|^2 \ .$$

Thus we conclude:

$$\begin{aligned}\widetilde{\phi}(\tilde{a}) &= \widetilde{\phi}(a + z\mathbb{1}) = \widetilde{\phi}(a) + \widetilde{\phi}(z\mathbb{1}) = \phi(a) + z\langle \psi \mid \psi \rangle \\ &= \langle \psi \mid \pi(a)\psi \rangle + \langle \psi \mid \pi(z\mathbb{1})\psi \rangle = \langle \psi \mid \pi(a + z\mathbb{1})\psi \rangle \\ &= \langle \psi \mid \pi(\tilde{a})\psi \rangle \ .\end{aligned}$$

In example 2.8.4 we have seen, that such a linear functional is positive (in this case on $\widetilde{A}$). □

Corollary 2.8.12.
Let B be a C^*-sub algebra of A. Every positive linear functional on B extends to a positive linear functional ψ on A with $\|\phi\| = \|\phi\|$. If B_+ is hereditary in A_+, then the extension is unique.

Proof 2.8.13.
Let $\overline{B} := C^*(B, \mathbb{1})$ be the C^*-algebra, induced by B and $\mathbb{1}$, where $\mathbb{1} \in \widetilde{A}$. It is clear, that $\overline{B} \subset \widetilde{A}$ as C^*-sub algebra. The prove of corollary 2.8.10 applies also for $\overline{B}$, such that there is a unique $\widetilde{\phi}$ on $\overline{B}$, that is positive. By the Hahn-Banach theorem, there is a linear extension $\widetilde{\psi}$ of $\widetilde{\phi}$ on $\widetilde{A}$ with $\|\widetilde{\psi}\| = \|\widetilde{\phi}\|$. Since $\widetilde{\phi}$ is positive on the unital C^*-algebra $\overline{B}$, it holds that:

$$\widetilde{\psi}(\mathbb{1}) = \widetilde{\phi}(\mathbb{1}) = \|\widetilde{\phi}\| = \|\widetilde{\psi}\| \ .$$

Hence by theorem 2.7.10 $\widetilde{\psi}$ is positive on $\widetilde{A}$. Then, the restriction $\psi := \widetilde{\psi}|_A$ is a positive linear extension of ϕ.

Assume now, that B_+ is hereditary, and let $\psi, \widetilde{\psi}, \phi$ and $\widetilde{\phi}$ as before. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of B, then:

$$\|\psi\| = \|\widetilde{\psi}\| = \widetilde{\psi}(\mathbb{1}) = \widetilde{\phi}(\mathbb{1}) = \|\widetilde{\phi}\| = \|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) \ .$$

Let $a \in A$ with $\|A\| \leq 1$, then $a^*a \in A_+$ with $\|a^*a\| \leq 1$. Because of lemma 2.4.11 it follows that $a^*a \leq \|a^*a\|\mathbb{1} \leq \mathbb{1}$ and with 2.4.13 we find

$$0 \leq (au_\lambda)^* au_\lambda = u_\lambda a^* a u_\lambda \leq u_\lambda^2 \in B_+ \ .$$

Since B_+ is hereditary, it follows that $u_\lambda b^* b u_\lambda \in B_+ \subset B$. Since $\langle A_+\rangle_{\mathbb{C}} = A$ and $\widetilde{a} = a + z\mathbb{1}$, it follows that $u_\lambda \widetilde{A} u_\lambda \subset B$. Using the Cauchy-Schwarz inequality (theorem 2.7.4) we see that (using $\widetilde{\psi}(b) = \psi(b) = \phi(b)$ for all $b \in B$):

$$\begin{aligned}
|\psi(a - u_\lambda a u_\lambda)| &= |\widetilde{\psi}(a - u_\lambda a u_\lambda)| = |\widetilde{\psi}((\mathbb{1} - u_\lambda)a + u_\lambda a(1 - u_\lambda))| \\
&\leq |\widetilde{\psi}((\mathbb{1} - u_\lambda)a)| + |\widetilde{\psi}(u_\lambda a(1 - u_\lambda))| \\
&\leq \sqrt{\widetilde{\psi}((\mathbb{1} - u_\lambda)^2)\widetilde{\psi}(a^*a)} + \sqrt{\widetilde{\psi}(a^* u_\lambda^2 a)\widetilde{\psi}((\mathbb{1} - u_\lambda)^2)} \\
&= \sqrt{\widetilde{\psi}(\mathbb{1} - u_\lambda)}\left(\sqrt{\widetilde{\psi}(a^*a)} + \sqrt{\widetilde{\psi}(a^* u_\lambda^2 a)}\right) \longrightarrow 0 \ ,
\end{aligned}$$

since by construction $\widetilde{\psi}(\mathbb{1}) = \|\psi\|$ and since ψ is positive, so $\lim_{\lambda\in\Lambda} \widetilde{\psi}(u_\lambda) = \lim_{\lambda\in\Lambda} \psi(u_\lambda) = \|\psi\|$. This shows that

$$\psi(a) = \lim_{\lambda\in\Lambda} \psi(u_\lambda a u_\lambda) = \lim_{\lambda\in\Lambda} \phi(u_\lambda a u_\lambda) \ .$$

This shows, that ψ is uniquely determined by ϕ. □

Definition 2.8.14.
For the set of states on A, we write $S(A)$. Furthermore, for every linear functional $\phi\colon A \to \mathbb{C}$ we define ϕ^* by

$$\phi^*(a) = \overline{\phi(a)} \qquad \forall a \in A \ .$$

The next two results involve the concept of convex hull. The convex hull of a subset $K \subset X$ can be defined most abstractly as the intersection of all convex set that contain K. A more ready definition is the set of convex combinations, that the convex hull of K is

$$\left\{ \sum_{k=1}^n \alpha_k c_k \,\middle|\, c_k \in K \ , \ \alpha_k \in [0,1] \ , \ \sum_{k=1}^n \alpha_k = 1 \ , \ n \in \mathbb{N} \right\} \ .$$

Lemma 2.8.15.
Let $Q = Q(A)$ be the set of all positive linear functionals on A with Norm ≤ 1.

Then the convex hull of $Q \cup -Q$ is

$$K \coloneqq \{\phi_1 - \phi_2 \mid \phi_1, \phi_2 \in Q\}$$

and is weak-$*$-compact.

In the proof we will also prove the following corollary

Corollary 2.8.16.
The set Q is convex.

Proof 2.8.17.
Let $B_1 = \{\phi \in A' \mid \|\phi\| \leq 1\}$ be the closed unit ball w.r.t. the operator norm. Consider the map $\mu_a \colon A' \to \mathbb{C}$, defined by $\mu_a(\phi) = \phi(a)$. Then by corollary 1.3.6 μ_a is continuous in the weak-$*$-topology. Thus:

$$Q = B_1 \bigcap_{a \in A_+} \mu_a^{-1}([0, \infty)) \ .$$

Since B is closed in the norm topology it is also closed in the weak-$*$-topology (see theorem 1.3.8). The preimage of a closed set, w.r.t. a continuous map is closed, as well as arbitrary intersections, it follows that Q is closed in the weak-$*$-topology. From the Banach-Alaoglu theorem 1.3.10 it follows that B_1 is compact in the weak-$*$-topology and thus Q is compact in the weak-$*$-topology.

Let $D = \{\phi_1 - \phi_2 \mid \phi_1, \phi_2 \in Q\}$. First we show that $K \subset D$. Since $0 \in Q$ it follows that $Q \cup -Q \subset D$. Using theorem 2.7.10 we find that for positive linear functionals it holds that:

$$\|\phi_1 + \phi_2\| = \lim_{\lambda \in \Lambda} (\phi_1 + \phi_2)(u_\lambda) = \lim_{\lambda \in \Lambda} \phi_1(u_\lambda) + \phi_2(u_\lambda) = \|\phi_1\| + \|\phi_2\|$$

for $\phi_1, \phi_2 \geq 0$. Then we see that Q is convex:

$$\|(1-t)\phi_1 + t\phi_2\| = (1-t)\|\phi_1\| + t\|\phi_2\| =\leq 1 - t + t = 1 \ .$$

Thus for $\phi_1, \phi_2, \psi_1, \psi_2 \in Q$ it holds that:

$$t(\phi_1 - \phi_2) + (1-t)(\psi_1 - \psi_2) = \underbrace{t\phi_1 + (1-t)\psi_1}_{\in Q} - \underbrace{t\phi_2 + (1-t)\psi_2}_{\in Q} \in D \ .$$

Hence D is also convex. Since the convex hull K is defined to be the smallest convex set, containing $Q \cup -Q$ it follows that $K \subset D$.

For the opposite inclusion, we want to show that $\phi_1 - \phi_2$ can be written as convex sum $\phi_1 - \phi_2 = t\psi_1 + (1+t)\psi_2 \in K$. Take $t = \frac{\|\phi_1\|}{\|\phi_1\| + \|\phi_2\|} \in [0,1]$, then:

$$t\phi_1 = \frac{\|\phi_1\|}{\|\phi_1\| + \|\phi_2\|}\phi_1 \in Q$$

$$\text{and} \qquad (1-t)(-\phi_2) = -(1-t)\phi_2 = -\frac{\|\phi_2\|}{\|\phi_1\| + \|\phi_2\|}\phi_2 \in -Q \ .$$

Next we observe that from the positive definiteness of the norm we have

$$\| \|\phi_2\|\phi_1 - \|\phi_1\|\phi_2\| = \|\phi_2\| \cdot \|\phi_1\| - \|\phi_1\| \cdot \|\phi_2\| = 0$$
$$\Rightarrow \quad \|\phi_2\|\phi_1 - \|\phi_1\|\phi_2 = 0$$

$$\begin{aligned} \Rightarrow \quad (\|\phi_1\| + \|\phi_2\|)(\phi_1 - \phi_2) &= \|\phi_1\|\phi_1 - \|\phi_1\|\phi_2 + \|\phi_2\|\phi_1 - \|\phi_2\|\phi_2 \\ &= \|\phi_1\|\phi_1 - \|\phi_2\|\phi_2 \ . \end{aligned}$$

Hence we find:

$$\begin{aligned} \phi_1 - \phi_2 &= \frac{\|\phi_1\| + \|\phi_2\|}{\|\phi_1\| + \|\phi_2\|}(\phi_1 - \phi_2) = \frac{\|\phi_1\|\phi_1 - \|\phi_2\|\phi_2}{\|\phi_1\| + \|\phi_2\|} \\ &= t\phi_1 + (1-t)(-\phi_2) \in C \ , \end{aligned}$$

which shows that $K \subset D$. Thus we have shown that $D = C$.

The theorem of Eberlin-Šmulian states that if $E \subset X$ is a non-empty subset of a Banach space, then if every sequence has a sub sequence that converges w.r.t. the weak-∗-topology (called sequentially compact), then E is compact in the weak-∗-topology. Since $K \subset A'$ we only need to show sequentially compactness.

Let $(\phi_n - \varphi_n)$ be a sequence in K. Since Q is weak-∗-compact, there are convergent sub sequences (ϕ_{k_n}) of (ϕ_n) and (φ_{ℓ_n}) of (φ_n). Then $(\phi_{k_n} - \varphi_{\ell_n})$ is a convergent sub sequence of $(\phi_n - \varphi_n)$.

□

Theorem 2.8.18.
The set $S = S(A)$ is convex. Furthermore, the set

$$\{\phi \in A' \mid \|\phi\| \leq 1 \ , \ \phi^* = \phi\}$$

is the convex hull of $S \cup -S$.

Proof 2.8.19.
For convexity of $S(A)$ we need to show, that for $\phi_1, \phi_2 \in S(A)$ it holds that $(1-t)\phi_1 + t\phi_2 \in S(A)$. As seen in the proof of lemma 2.8.15 it holds that

$$\|\phi_1 + \phi_2\| = \|\phi_1\| + \|\phi_2\| \qquad \phi_1, \phi_2 \geq 0 \ ,$$

and hence:

$$\|(1-t)\phi_1 + t\phi_2\| = (1-t)\|\phi_1\| + t\|\phi_2\| = 1 - t + t = 1 \ .$$

For the second claim we want to show that the convex hull J of $S \cup -S$ is the convex hull K of $Q \cup -Q$. Since $S \cup -S \subset Q \cup -Q$ it is enough to show that

$K \subset J$. Since the convex hull is the smallest set convex set, containing the set in question, it is enough to show $Q \cup -Q \subset J$. Let $\phi \in Q$, then $\psi = \frac{\phi}{\|\phi\|} \in S$ and $\phi = \|\phi\|\psi$. Let now $\alpha = \frac{1-\|\phi\|}{2}$, then

$$\phi = \|\phi\|\psi + \alpha\varphi + \alpha(-\varphi) \quad \forall\, \varphi \in S \ .$$

This is a convex sum, such that $\phi \in J$. The same construction works for $-\phi \in -Q$ and $-\psi \in -S$, such that $Q \cup -Q \subset J$. Hence the convex hull of $S \cup -S$ is $K = \{\phi_1 - \phi_2 \mid \phi_1, \phi_2 \in Q\}$. From lemma 2.8.15 it follows that it is weak-$*$-compact.

Let $a \in A_{sa}$ and $B := C^*(a)$. Because of corollary 2.3.6, there is a $\varphi_z \in \Gamma_B$ for all $z \in \sigma(a) \setminus \{0\}$ with $\varphi_z(a) = z$. The extension to $\widetilde{B}$ satisfies $\widetilde{\varphi}_z(\mathbb{1}) = 1$ by lemma 2.2.6. Since $\widetilde{B}$ is unital, theorem 2.7.10 together with lemma 1.4.17 show that $\widetilde{\varphi}_z$ is a positive linear functional and so is φ_z. Then there is a unique extension (corollary 2.8.12) χ_z of φ_z to A, since $B \subset A$ is a C^*-sub algebra. Corollary 2.8.10 also states that $\|\varphi_z\| = \|\widetilde{\varphi}_z\| = 1$, such that $\|\chi_z\| = \|\varphi_z\| = 1$ by corollary 2.8.12. Hence $\chi_z \in S$. It follows that

$$\begin{aligned}\|a\| = \rho(a) = \sup\{z \in \sigma(a)\} &= \sup\{\varphi(a) \mid \varphi \in \Gamma_A\} \\ &\leq \sup\{|\phi(a)| \mid \varphi \in S\} \leq \|\varphi\|\|a\| = \|a\| \ .\end{aligned}$$

Assume now, that there is a $\psi \in A' \setminus K$ with $\psi^* = \psi$ and $\|\psi\| \leq 1$. By the Hahn-Banach theorem there is a $\varepsilon > 0$ and an $a \in A_{sa} = ((A_{sa})'_\sigma)'$ such that $\psi(a) > \varepsilon$ but $\phi(a) \leq \varepsilon$ for all $\phi \in K$, since K is weak-$*$-compact. Since $K = -K$ it follows that $|\phi(a)| \leq \varepsilon$ for all $\phi \in K$ and thus:

$$\begin{aligned}\|a\| &\leq \sup\{|\phi(a)| \mid \varphi \in S\} \leq \sup\{|\phi(a)| \mid \varphi \in K\} \\ &\leq \varepsilon < \psi(a) \leq \|\psi\|\|a\| \leq \|a\| \ ,\end{aligned}$$

which is a contradiction. Hence $K = \{\phi \in A' \mid \|\phi\| \leq 1 \ , \ \phi^* = \phi\}$. □

Definition 2.8.20.
Let $F \subset S(A)$ be a subset. It is called **separating**, if

$$\forall\, a \in A_+ \colon (\forall\, \phi \in F \colon \phi(a) = 0) \quad \Rightarrow \quad a = 0 \ .$$

To prove the next corollary, we need a statement similar to corollary 2.1.5.

Corollary 2.8.21.
Let A be a C^*-algebra, then every linear functional ϕ can be written as sum $\phi = \psi + i\psi'$ with self adjoint linear functionals ψ, ψ'.

Proof 2.8.22.
It holds that

$$\phi = \frac{1}{2}(\phi + \phi^*) + i\frac{1}{2i}(\phi - \phi^*).$$

Furthermore for $\psi = \frac{1}{2}(\phi + \phi^*)$ and $\psi' = \frac{1}{2i}(\phi - \phi^*)$. It is straightforward to show that $\psi^* = \psi$ and $\psi'^* = \psi'$ by plugging in the definitions. □

Corollary 2.8.23.
The set of states $S = S(A)$ is separating. Furthermore, if A is a separable C^*-algebra, then there is a $\phi \in S$, such that $\{\phi\} \subset S$ is separating.

Proof 2.8.24.
Let $a \in A_+$ such that $\phi(a) = a$ for all $\phi \in S$. Then it also holds that $-\phi(a) = 0$, i.e. $\psi(a) = 0$ for all $\psi \in -S$. In the proof of theorem 2.8.18 we have seen that the convex hull of $S \cup -S$ is

$$K = \{\phi_1 - \phi_2 \mid \phi_1, \phi_2 \in Q\}f = \{\phi \in A' \mid \|\phi\| \leq 1 \ , \ \phi^* = \phi\} \ .$$

This shows that $\phi(a) = 0$ for all $\phi \in K$. Put differently (and rescaling), for all self adjoint linear functionals $\phi \in A'$ it holds that $\phi(a) = 0$. From corollary 2.8.21 it follows that $\phi(a) = 0$ for all $\phi \in A'$. But then $a = 0$, which is the first claim.

With results and methods from functional analysis and topology, it can be shown that there is a dense sequence $\{\phi_n\}$ in S. Consider

$$\phi := \sum_{n=0}^{\infty} 2^{-(n+1)} \phi_n \ \in S \ .$$

Since the sequence is dense, the subset $\{\phi_n \mid n \in \mathbb{N}\} \subset S$ is separating. If $\phi(a) = 0$ for all $a \in A_+$, then $\phi_n(a) = 0$ for all $n \in \mathbb{N}$. Then, since $\{\phi_n \mid n \in \mathbb{N}\}$ is separating, $a = 0$, showing that $\{\phi\}$ is separating. □

Theorem 2.8.25.
For every $C*$-algebra there is a non-degenerate faithful $*$-representation, called ***universal $*$-representation****. If A is separable, then one can assume that the representation space is separable.*

Proof 2.8.26.
Let $F \subset S$ be a separating subset (existence ensured by corollary 2.8.23) and define

$$\pi := \bigoplus_{\phi \in F} \pi_\phi \ .$$

By theorem 2.8.8 the representations π_ϕ are non-degenerate (and thus is π on $\mathcal{H} = \bigoplus_{\phi \in F} \mathcal{H}_\phi$) and $\phi(a) = \langle \psi_\phi \mid \pi_\phi(a)\psi_\phi \rangle$ for all $a \in A$.

Let $a \in \mathrm{Ker}(\pi)$, then it holds that $a \in \mathrm{Ker}(\pi_\phi)$ for all $\phi \in F$ and thus

$$\phi(a) = \langle \psi_\phi \mid \pi_\phi(a)\psi_\phi \rangle = \langle \psi_\phi \mid \pi_0 \psi_\phi \rangle = 0 \ .$$

If a is positive, then because F is separating it follows that $a = 0$. Since every element in A can be written as linear combination of positive elements, it follows that $\mathrm{Ker}(\pi) = 0$, proving that π is faithful.

Assume now, that A is separable, then by corollary 2.8.23 one can choose $F = \{\phi\}$. By the construction of the GNS-representation $\pi = \pi_\phi$ and more importantly, A/N is dense in $\mathcal{H}$. But since A is separable, so is $\mathcal{H}$. □

Corollary 2.8.27.
Every C^*-algebra is isometrically $*$-isomorphic to a C^*-sub algebra of $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. If A is separable, then one can assume that $\mathcal{H}$ is separable.

Proof 2.8.28.
By theorem 2.8.25 there is a faithful $*$-representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$, i.e. an injective $*$-morphism. By theorem 2.6.10 the image of a $*$-morphism is a C^*-sub algebra, i.e. $\pi(A) \subset \mathcal{L}(\mathcal{H})$ is a C^*-sub algebra. Also, $\pi\colon A \to \pi(A)$ is surjective, injecive and by theorem 2.6.10 also isometric. Hence an isometrical-$*$-isomorphism.

By theorem 2.8.25, if A is separable, then one can assume $\mathcal{H}$ to be separable. □

2.9 Von Neumann algebras

In this section, $\mathcal{H}$ denotes a Hilbert space.

2.9.1 Definition of von Neumann algebras

Definition 2.9.1.
The **weak operator topology** (WOT) on $\mathcal{L}(\mathcal{H})$ is the locally convex topology defined by the semi norms:

$$A \longmapsto |\langle \phi | A\psi \rangle| \quad \forall\, \phi, \psi \in \mathcal{H}\ .$$

The **strong operator topology** (SOT) on $\mathcal{L}(\mathcal{H})$ is the locally convex topology defined by the semi norms:

$$A \longmapsto \|A\psi\| \quad \forall\, \psi \in \mathcal{H}\ .$$

Using the properties of nets, we can prove the following corollary:

Corollary 2.9.2.
The WOT is weaker than the SOT and the SOT is weaker than the norm topology.

Proof 2.9.3.
Let V be an open set in WOT. Let $A_0 \in V$, then there is $\varepsilon > 0$ and $F =$

$\{(\phi_1, \psi_1), \dots, (\phi_n, \psi_n)\}$, such that $B^{WOT}_{F,\varepsilon}(A_0) \subset V$. For $A \in B^{WOT}_{F,\varepsilon}(A_0)$ this means that

$$|\langle \phi \mid (A - A_0)\psi \rangle| < \varepsilon \forall\ \phi, \psi \in F\ .$$

The case of $\phi_k = 0$ for all k is trivial, so define $\delta := \max_k\{\|\phi_k\|\} > 0$ and $F' = \{\psi_1, \dots, \psi_n\}$. Then for $B \in B^{SOT}_{F',\frac{\varepsilon}{\delta}}(A_0)$ it follows that for all $\phi \in F$ and $\psi \in F'$:

$$|\langle \phi \mid (B - A_0)\psi \rangle| \leq \|\phi\| \|(B - A_0)\psi\| < \|\phi\| \frac{\varepsilon}{\delta} \leq \varepsilon\ ,$$

showing that $B^{SOT}_{F',\frac{\varepsilon}{\delta}}(A_0) \subset B^{WOT}_{F,\varepsilon}(A_0) \subset V$. Hence V is SOT open.

Showing that SOT is weaker than the norm topology is exactly the same, using that:

$$\|(B - A_0)\psi\| \leq \|\psi\| \|B - A_0\|\ .$$

□

Corollary 2.9.4.

i) The product $(A, B) \to AB$ is SOT continuous on $\mathcal{B} \times \mathcal{H}$ for every norm bounded subset $\mathcal{B} \subset \mathcal{H}$.

ii) For fixed B the maps $A \mapsto AB$ and $A \mapsto BA$ are WOT continuous.

iii) The involution $*$ is WOT continuous as map $*\colon (\mathcal{L}(\mathcal{H}), WOT) \to (\mathcal{L}(\mathcal{H}), WOT)$.

Proof 2.9.5.

i) Let $\mathcal{O}$ be open and $(A_0, B_0) \in \mathcal{O}' := \{(A, B) \in \mathcal{B} \times \mathcal{L}(\mathcal{H}) \mid AB \in \mathcal{O}\}$. Let $F = \{\psi_1, \dots, \psi_n\}$ and $\varepsilon > 0$, such that $B^{\mathrm{SOT}}_{F,\varepsilon}(A_0B_0) \subset \mathcal{O}$ and define $s := \sup\{A \in B^{\mathrm{SOT}}_{B_0F,\frac{\varepsilon}{2}}(A_0)\}$. Then for $A \in B^{\mathrm{SOT}}_{B_0F,\frac{\varepsilon}{2}}(A_0)$ and $B \in B^{\mathrm{SOT}}_{F,\frac{\varepsilon}{2s}}(B_0)$ it follows that

$$B^{\mathrm{SOT}}_{F,\frac{\varepsilon}{2}}(A_0) \times B^{\mathrm{SOT}}_{B_0F,\frac{\varepsilon}{2}}(B_0) \subset \mathcal{O}$$

since $AB \in B^{\mathrm{SOT}}_{F,\varepsilon}(A_0B_0)$ for all such A and B, because of

$$\begin{aligned} \|(AB - A_0B_0)\psi\| &= \|A(B - B_0)\psi + (A - A_0)B_0\psi\| \\ &\leq \|A\| \cdot \|(B - B_0)\psi\| + \|(A - A_0)B_0\psi\| \\ &< \|A\| \frac{\varepsilon}{2s} + \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

for all $\psi \in F$.

ii) Œ for the map $A \mapsto AB$. Chose $A \in B^{\mathrm{WOT}}_{\{(\phi,B\psi)\},\varepsilon}(A_0)$, then:

$$|\langle \phi \mid (AB - A_0B)\psi \rangle| = |\langle \phi \mid (A - A_0)B\psi \rangle| \leq \varepsilon\ .$$

The rest is similar to i).

[4] Let $\tau_1 \subset \tau_2$, then every open set in τ_1 is also open is τ_2. Since a set is closed, if its complement is open, the same applies to closed sets.

iii) Let $A \in B^{\text{WOT}}_{\{(\psi,\phi)\},\varepsilon}(A_0)$, then $A^* \in B^{\text{WOT}}_{\{(\phi,\psi)\},\varepsilon}(A_0^*)$, since:

$$\begin{aligned} |\langle \phi \mid (A^* - A_0^*)\psi\rangle| &= |\langle \phi \mid (A - A_0)^*\psi\rangle| = |\langle (A - A_0)\phi \mid \psi\rangle| \\ &= |\langle \psi \mid (A - A_0)\phi\rangle| < \varepsilon \ . \end{aligned}$$

Again, the rest is the same as i).

□

Lemma 2.9.6.
Let $(A_\lambda)_{\lambda\in\Lambda}$ be a monotonously increasing net of positive elements in $\mathcal{L}(\mathcal{H})$, that is bounded w.r.t. the norm. Then $(A_\lambda)_{\lambda\in\Lambda}$ converges in SOT.

Proof 2.9.7.
Œ let $\|A_\lambda\| \leq 1$. Let $\psi \in \mathcal{H}$ and $\phi\colon \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ defined by

$$\phi(A) := \langle \psi \mid A\psi\rangle \ , \qquad \forall\ A \in \mathcal{L}(\mathcal{H}) \ .$$

Then, by example 2.8.4 ϕ is a positive linear functional. Since (A_λ) is monotonously increasing and bound by $\|A_\lambda\| \leq 1$, the net $\phi(A_\lambda)$ converges against an $r \geq 0$. Comparing the proof of example 2.8.4, we see that for $\lambda \leq \mu$ it holds that (also using corollary 2.5.8 and 2.7.2):

$$\|(A_\mu - A_\lambda)\psi\|^2 = \phi((A_\mu - A_\lambda)^2) \leq \phi(A_\mu - A_\lambda) \longrightarrow r - r = 0 \ .$$

This shows that $A_\lambda\psi$ is a Cauchy net and hence converges against an $A\psi \in \mathcal{H}$. Thus (A_λ) converges against A in SOT. □

Theorem 2.9.8.
Let Φ be a linear functional on $\mathcal{L}(\mathcal{H})$, then the following claims are equivalent:

i) There are $\psi_k, \phi_k \in \mathcal{H}$ for $k = 1, \ldots, n$, such that

$$\Phi(A) = \sum_{k=1}^{n} \langle \phi_k \mid A\psi_k\rangle \qquad \forall\ A \in \mathcal{L}(\mathcal{H}) \ .$$

ii) Φ is WOT continuous.

iii) Φ is SOT continuous.

Remark 2.9.9.
As a remainder. The representation theorem of Riesz states that for every $A \in \mathcal{L}(\mathcal{H})$ there is a unique $\psi \in \mathcal{H}$, such that $A = \langle \psi \mid \cdot\rangle$. As a result, H^* is again a Hilbert space and $\mathcal{H}^{**} = \mathcal{H}$.

Proof 2.9.10.

i) ⇒ ii)
Let $A_0 \in \mathcal{O} := \{A \in \mathcal{L}(\mathcal{H}) \mid |\Phi(A) - \Phi(A_0)| < \varepsilon\}$. Choose $F = \{(\phi_k, \psi_k)\}$, such that $A \in B^{\mathrm{WOT}}_{F,\varepsilon}(A_0)$, i.e. $Z = A - A_0 \in B^{\mathrm{WOT}}_{F,\frac{\varepsilon}{n}}$, then:

$$|\Phi(A) - \Phi(A_0)| = |\Phi(A - A_0)| = |\Phi(z)| = \left|\sum_{k=1}^{n} \langle \phi_k \mid A\psi_k \rangle\right|$$
$$\leq \sum_{k=1}^{n} |\langle \phi_k \mid A\psi_k \rangle| < \sum_{k=1}^{n} \frac{\varepsilon}{n} \leq \varepsilon \ .$$

The rest is standard procedure.

ii) ⇒ iii)
Since $\Phi\colon \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ is WOT continuous and SOT is a finer topology it follows immediately.

iii) ⇒ i)
There are $\psi_1, \dots, \psi_n \in \mathcal{H}$, such that (in fact it holds for all $\phi_1, \dots, \phi_n$)

$$|\Phi(A)| \leq \max_{k=1,\dots,n} \|A\psi_k\| \leq \sqrt{\sum_{k=1}^{n} \|A\psi_k\|^2} := \|\pi(A)\psi\|_{\mathcal{K}},$$

where we defined $\mathcal{K} := \mathcal{H}^n = \mathcal{H} \oplus \dots \oplus \mathcal{H}$, $\psi := \psi_1 \oplus \dots \oplus \psi_n$ and $\pi(A)\xi := A\xi_1 \oplus \dots \oplus A\xi_n$ for all $\xi \in \mathcal{K}$.

Let $V := \pi(\mathcal{L}(\mathcal{H}))\psi \subset \mathcal{K}$ and define $\varphi\colon V \to \mathbb{C}$ by

$$\varphi(\pi(A)\psi) := \Phi(A) \qquad \forall\, A \in \mathcal{L}(\mathcal{H}) \ .$$

φ extends to the closure $\overline{V}$, since for all $\xi = \pi(A)\psi \in V$ it holds that

$$|\varphi(\xi)| = |\Phi(A)| \leq \|\pi(A)\psi\|_{\mathcal{K}} = \|\xi\|_{\mathcal{K}} \ .$$

Because of the Riesz representation theorem, there is $\xi = \xi_1 \oplus \dots \oplus \xi_n \in \mathcal{K}$, such that $\varphi(\pi(A)\psi) = \langle \xi \mid \pi(A)\psi \rangle$. Hence

$$\Phi(A) = \varphi(\pi(A)\psi) = \langle \xi \mid \pi(A)\psi \rangle = \sum_{k=1}^{n} \langle \xi_k \mid A\psi_A \rangle \ .$$

□

Remark 2.9.11.
The map $\pi\colon \mathcal{L}(\mathcal{H}) \to \mathbb{L}(\mathcal{H}^n)$ is a $*$-representation.

Corollary 2.9.12.
Let $K \subset \mathcal{L}(\mathcal{H})$ be convex. K is SOT closed, if and only if K is WOT closed.

For the proof we will use [Mur90, proof of theorem 4.2.7]

Proof 2.9.13.
Assume that K is SOT closed. For all A in the WOT closure, there is a net $(A_\lambda)_{\lambda\in\Lambda}$, converging in WOT against A. Hence, for every WOT continuous linear functional Φ on $\mathcal{L}(\mathbb{H})$ it holds that $\Phi(A) = \lim_\lambda \Phi(A_\lambda)$. By theorem 2.9.8 Φ is also SOT continuous. Thus A is in the strong closure of K, i.e. $A \in K$. Hence K is WOT closed.

The opposite direction follows from corollary 2.9.2. □

Definition 2.9.14.
Let $M \subset \mathcal{L}(\mathcal{H})$, then the **commutant** M^c of M is defined by:

$$M^c := \{A \in \mathcal{L}(\mathcal{H}) \mid \forall\ B \in M : AB = BA\} \ .$$

The **bicommutant** is $M^{cc} = (M^c)^c$.

A first observation is, that $M \subset M^{cc}$. This can be seen as follows. Assume $A \in M$. By definition $B \in M^c$ means that $BA = AB$, such that $AB = BA$ for all $B \in M^c$. Hence $A \in M^{cc}$.

Another direct observation is, that for $M_1 \subset M_2$ it follows that $M_2^c \subset M_1^c$. Both observations together yield

$$M^c \subset (M^c)^{cc} = M^{ccc}\ ,\ M \subset M^c \ \Rightarrow\ M^c \subset M^{ccc} \quad \Rightarrow \quad M^c = M^{ccc}\ .$$

Corollary 2.9.15.
Let $M \subset \mathcal{L}(\mathcal{H})$, then M^c is a WOT closed sub algebra. If $M = M^*$, then M^c is a C^*-algebra.

Proof 2.9.16.
That M^c is a sub algebra follows from a direct check of the algebra axioms. It remains to show that it is closed in the WOT. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a net in M that converges w.r.t. the WOT against A. Since the product for a fixed $B \in \mathcal{L}(\mathcal{H})$ is continuous, it follows that

$$BA = \lim_{\lambda\in\Lambda} BA_\lambda = \lim_{\lambda\in\Lambda} A_\lambda B = AB\ .$$

Hence $A \in M^c$, showing that M^c is closed w.r.t. the WOT.

For the last claim, it has to be shown that $A^* \in M^c$ if $A \in M^c$. Since by assumption $B^* = B$ for all $B \in M$ it follows that:

$$A^*B = A^*B^* = (BA)^* = (AB)^* = B^*A^* = BA^*\ .$$

□

Corollary 2.9.17.
Let $A \subset \mathcal{L}(\mathcal{H})$ and M its WOT closure. Then A^c is the WOT closure of M^c.

Proof 2.9.18.
Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in M^c. We need to show that $x = \lim_{\lambda\in\Lambda} x_\lambda \in A^c$. Let $a \in A \subset M$, then $x_\lambda a = a x_\lambda$. By corollary 2.9.4, the product with a is continuous, such that

$$xa = \lim_{\lambda\in\Lambda} x_\lambda a = \lim_{\lambda\in\Lambda} a x_\lambda = a \lim_{\lambda\in\Lambda} x_\lambda = ax \ .$$

Hence $x \in A^c$. □

To proof the next corollary, that will be used in the proof of the central theorem of this subsection, the concept of orthogonal projections is used, making it worthwhile to recall some properties.

Remark 2.9.19.
Let $U \subset \mathcal{H}$ be a closed sub vector space and $U^\perp$ its orthogonal complement. Then for every $\psi \in \mathcal{H}$, there exist unique $\phi \in U$ and $\phi^\perp \in U^\perp$, such that $\psi = \phi + \phi^\perp$. Then there is a unique linear operator, defined by $p(\psi) = \phi$ with $\mathrm{im}(p) = U$ and $\mathrm{Ker}(p) = U^\perp$. This operator is called **(orthogonal) projection**. It is characterized by $p^2 = p$ and $p^* = p$, since:

$$\langle p(\psi) \mid \psi' \rangle = \langle p(\psi) \mid \phi' \rangle + \langle p(\psi) \mid \phi'^\perp \rangle = \langle \phi \mid \phi' \rangle = \ldots = \langle \psi \mid p(\psi') \rangle \ .$$

Corollary 2.9.20.
Let $M \subset \mathcal{L}(\mathcal{H})$ be a C^*-algebra. Let $B \in M^{cc}$, $\psi \in \mathcal{H}$ and $\varepsilon > 0$. Then there is an $A \in M$, such that

$$\|(B - A)\psi\| \leq \varepsilon \ .$$

Proof 2.9.21.
Define $\mathcal{K} := \overline{M\psi}$. Then $\mathcal{K}$ is M-invariant by definition. Let $\mathcal{K}^\perp$ be the orthogonal complement, then $\mathcal{K}^\perp$ is also M invariant, since

$$\langle M\mathcal{K}^\perp \mid \mathcal{K} \rangle = \langle \mathcal{K}^\perp \mid M^*\mathcal{K} \rangle = \langle \mathcal{K}^\perp \mid M\mathcal{K} \rangle = \langle \mathcal{K}^\perp \mid \mathcal{K} \rangle = 0 \ .$$

Let p be the projection on $\mathcal{K}$. Because of the M invariance, it holds for all $A \in M$ that:

$$pAp = Ap \qquad \text{and} \qquad pA(\mathbb{1} - p) = 0 \ .$$

Hence:

$$Ap = pAp + pA(\mathbb{1} - p) = pA \ ,$$

so $p \in M^c$. For $B \in M^{cc}$ it holds $pB = Bp$ showing that $\mathcal{K} \ni pB\psi = Bp\psi = B\psi$ and thus $B\psi \in \mathcal{K}$. Let $(\psi_n)_n$ be a sequence in $M\psi$ that converges against $B\psi$

(existence, because $\mathcal{K}$ is defined to be Norm closed). Then every ψ_n can be written as $A_n\psi$ for an $A_n \in M$. It follows that

$$\lim_{n\to\infty} A_n\psi = B\psi.$$

Since this limit is defined in the norm topology, for every $\varepsilon > 0$ there is an $A \in M$ with

$$\|(B - A)\psi\| \leq \varepsilon \ .$$

□

Theorem 2.9.22 (**Von Neumann bicommutant theorem**).
Let $M \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra that contains $\mathbf{1} = \mathbb{1}$, then the following properties are equivalent:

i) $M = M^{cc}$.

ii) M *is WOT closed.*

iii) M *is SOT closed.*

Proof 2.9.23.

ii) ⇔ iii)
Since M is an algebra, it is convex, such that ii) ⇔ iii) because of corollary 2.9.12.

i) ⇒ ii)
Applying corollary 2.9.15 to M^c, it follows that M^{cc} is WOT closed. From i) it follows that M is WOT closed.

iii) ⇒ i)
Let π be the $*$-representation onto $\mathcal{H}^n$ from the proof of theorem 2.9.8. The space $\mathcal{L}(\mathcal{H}^n)$ consists of matrices with linear operators as coefficients:

$$\mathcal{L}(\mathcal{H}^n) = \{X = (X_{ij}) \mid i, j = 1, \ldots, n \ , \ X_{ij} \in \mathcal{L}(\mathcal{H})\} \ .$$

Let $X = (X_{ij}) \in \mathcal{L}(\mathcal{H}^n)$ and $A \in M$, then:

$$\pi(A)X - X\pi(A) = (AX_{ij} - X_{ij})$$

$$\Rightarrow \qquad \pi(M)^c = \{X = (X_{ij}) \mid X_{ij} \in M^c \quad \forall\, i, j\} \ .$$

$$\Rightarrow \qquad \pi(M)^{cc} = \{X = (X_{ij}) \mid X_{ij} \in M^{cc} \quad \forall i, j\} \ .$$

Since $\pi(M^{cc}) = \{\text{diag}(X_1, \ldots, X_n) \mid X_i \in M^{cc} \ , \ \forall\, i\}$ it follows that $\pi(M^{cc}) \subset \pi(M)^{cc}$.

Let $B \in M^{cc}$ and $\psi_1, \ldots, \psi_n \in \mathcal{H}$ and define $\Psi = \psi_1 \oplus \ldots \oplus \psi_n \in \mathcal{H}^n$. Corollary 2.9.20 shows, that for all $\varepsilon > 0$ there is an $A \in M$, such that

$$\begin{aligned} \varepsilon \geq &\|(\pi(B) - \pi(A))\Psi\| = \sqrt{\sum_{k=1}^{n} \|(B-A)\psi_k\|} \\ &= \sqrt{\sum_{k=1}^{n} \langle (B-A)\psi_k \mid (B-A)\psi_k \rangle} \end{aligned}$$

$$\Rightarrow \quad \sum_{k=1}^{n} \langle (B-A)\psi_k \mid (B-A)\psi_k \rangle \coloneqq \sum_{k=1}^{n} |\langle \phi_k \mid (B-A)\psi_k \rangle| \leq \varepsilon^2 \,.$$

This shows that B is in the WOT closure of M. Then it is in the SOT closure of M because ii) $\Leftrightarrow$ iii). By assumption M was already SOT closed, such that $B \in M$. Since $B \in M^{cc}$ was arbitrary, it follows that $M^{cc} \subset M$. Since $M \subset M^{cc}$ is always true, it follows that $M = M^{cc}$. □

Definition 2.9.24.
A C^*-sub algebra $M \subset \mathcal{L}(\mathcal{H})$, that satisfies the equivalent properties of theorem 2.9.22 is called **von Neumann algebra** or ***W**-algebra**.

2.9.2 Kaplansky density theorem

Before we can proof the Kaplansky density theorem, we need the concept of strongly continuous functions.

Definition 2.9.25.
A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is called **strongly continuous**, if for every net $(x_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{L}(\mathcal{H})_{sa}$ that converges to $x \in \mathcal{L}(\mathcal{H})_{sa}$ in SOT, it holds that $f(x_\lambda)$ converges to $f(x)$ in SOT.

As a reminder, $f(x_\lambda)$ refers to the functional calculus, where $\mathcal{L}(\mathcal{H})$ is the C^*-algebra.

Lemma 2.9.26.
Let $f\colon \mathbb{R} \to \mathbb{C}$ be continuous function, such that $f(0) = 0$ and there are $\alpha, \beta > 0$, such that $|f(t)| \leq \alpha|t| + \beta$ for all $t \in \mathbb{R}$. Then f is strongly continuous.

Proof 2.9.27.
Let S denote the set of all strongly continuous functions $\mathbb{R} \to \mathbb{C}$ and $S^b \subset S$ denote the subset of bounded functions. Since the product is SOT continuous (see corollary 2.9.4), it follows that $S^b S \subset S$. This also yields, that $(S^b, \|\cdot\|_\infty)$ is a commutative Banach algebra.

Let $e(t) = (1+t^2)^{-1}t$ and (x_λ) be a net in$\mathcal{L}(\mathcal{H})_{sa}$, that converges against $x \in \mathcal{L}(\mathcal{H})_{sa}$ in SOT. It holds that $\|(1+x_\lambda^2)^{-1}\| \leq 1$ and also $\|(1+x^2)^{-1}\| \leq 1$. Hence, using again, that the product is SOT continuous:

$$\begin{aligned}
&\|e(x_\lambda) - e(x)\psi\| = \|(1+x_\lambda^2)^{-1}x_\lambda\psi - (1+x^2)^{-1}x\psi\| \\
&\quad = \|(1+x_\lambda^2)^{-1}\left(x_\lambda(1+x^2) - (1+x_\lambda^2)x\right)(1+x^2)^{-1}\psi\| \\
&\quad = \|(1+x_\lambda^2)^{-1}(x_\lambda - x)(1+x^2)^{-1}\psi + (1+x_\lambda^2)^{-1}x_\lambda(x-x_\lambda)x(1+x^2)^{-1}\psi\| \\
&\quad \leq \|(1+x_\lambda^2)^{-1}\|\|(1+x^2)^{-1}\|\left(\|(x-x_\lambda)\psi\| + \|x_\lambda(x-x_\lambda)x\psi\|\right) \\
&\qquad \leq \|(x-x_\lambda)\psi\| + \|x_\lambda(x-x_\lambda)x\psi\| \longrightarrow 0
\end{aligned}$$

This shows that $e(x) = \lim_{\lambda\in\Lambda} e(x_\lambda)$ in SOT and so $e \in S^b$. The same holds for e_ε, defined by $e_\varepsilon(x) := e(\varepsilon x)$, as long as $\varepsilon > 0$. The functions $\{e_\varepsilon \mid \varepsilon > 0\}$ separate the points of $\mathbb{R} \setminus \{0\}$. From corollary 1.5.4 it follows that $C_0((R) \setminus \{0\}) \subset S^b$.

Let now f be as specified in the claim and denote $x = \mathbb{1}_\mathbb{R}(x)$. Since f is maximal of linear order by assumption, it holds that:

$$f \cdot (1+x^2)^{-1} \in C_0(\mathbb{R} \setminus \{0\}) \subset S^b \ .$$

Since $x \in S$, it holds that $f \cdot (1+x^2)^{-1}x \in S$. This function is still bounded, as denominator and nominator are both of quadratic order, such that $f \cdot (1+x^2)^{-1}x^2 \in S$. It follows that

$$f = f \cdot (1+x^2)^{-1}x + f \cdot (1+x^2)^{-1}x^2 \in S \ ,$$

which was the claim. □

Theorem 2.9.28 (**Kaplansky density theorem**).
Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra with SOT closure M. Then the unit ball $B(A)$ of A is dense is SOT dense in the unit ball $B(M)$ of M.
Furthermore, $B(A_{sa})$ is SOT dense in $B(M_{sa})$ and $B(A_+)$ is SOT dense in $B(M_+)$. If $\mathbb{1} = \mathbb{1}_\mathcal{H} \in A$, then $U(A)$, denoting the subset of unitary elements, is SOT dense in $U(M)$.

Proof 2.9.29.
Since A_{sa} is a convex cone in A, by theorem 2.4.9, the SOT closure and WOT closure are the same (corollary 2.9.12). By corollary 2.9.4, the involution $*$ is WOT continuous and hence the WOT closure of A_{sa} is M_{sa}. Consider the the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by

$$f(t) := \max(\min(t,1), -1) \ .$$

Then, by lemma 2.9.26, f is strongly continuous. Also note, that $f = \mathrm{Id}$ on $[-1,1]$. Let $x \in B(M_{sa})$, then there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in A_{sa}, such that $x_\lambda \to x$ in SOT, and thus

$$x = f(x) = \lim_{\lambda\in\Lambda} f(x_\lambda) \ .$$

Also, it holds that $f(x_\lambda) \in B(A_{sa})$. Hence, $(f(x_\lambda))_{\lambda\in\Lambda}$ is a net in $B(A_{sa})$, converging against $x \in B(M_{sa})$ in SOT, showing that the SOT closure of $B(A_{sa})$ is $B(M_{sa})$. That is, $B(A_{sa})$ is dense in $B(M_{sa})$.

Considering the function

$$f(t) := \max(\min(t, 1), 0) \ ,$$

the same steps show, that $B(A_+)$ is dense in $B(M_+)$.

Let $\mathbb{1} \in A$ and $u \in U(A)$. From the spectral theorem it follows that there is an $x \in M_{sa}$, such that $u = \exp(ix)$. Let (x_λ) be a net, that converges against x in SOT. Then, because of lemma 2.9.26, $u_\lambda = \exp(ix_\lambda)$ converges against $u = \exp(ix)$ in SOT. Since $u_\lambda = \exp(ix_\lambda)$, it is a net in $U(A)$, converging against $x \in U(M)$. Hence $U(A)$ is dense in $U(M)$.

Let $x \in B(M)$. The SOT closure of $M_2(A) \subset M_2(\mathcal{L}(\mathcal{H})) := \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ is $M_2(M)$. It holds that

$$y := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in B(M_2(M)_{sa}),$$

hence there is a net (y_λ) in $B(M_2(A)_{sa})$ converging against $y \in B(M_2(M)_{sa})$ in SOT. Taking the component $x_\lambda := (y_\lambda)_{12}$ it follows that $x_\lambda \in B(A)$ and $x_\lambda \to x$ in SOT. □

2.9.3 Partial isometries, projections and polar decomposition

Definition 2.9.30.
Let A be a C^*-algebra and $u \in A$. Then u is called **partial isometry**, if u^*u is a projection, i.e. $(u^*u)^2 = u^*u$
($(u^*u)^* = u^*u$ is always true).

For partial isometries, one finds that:

Lemma 2.9.31.
Let A be a C^*-algebra, $p := a^*a$ and $q := aa^*$ for an $a \in A$. Then the following properties are equivalent:

i) It holds that $p^2 = p$.

ii) It holds that $aa^*a = a$.

iii) It holds that $a^*aa^* = a^*$

iv) It holds that $q^2 = q$.

Proof 2.9.32.

ii) ⇔ iii) Applying the involution $*$.

ii) ⇔ i) The direction ii) ⇒ i) is apparent. For the opposite direction let $x = aa^*a - a$, then it holds that:

$$\begin{aligned} x^*x &= (a^*aa^*)(aa^*a) - (a^*aa^*)a - a^*(aa^*a) + a^*a \\ &= p^3 + p^2 + p^2 - p = 0 \ . \end{aligned}$$

From the C^*-norm identity $0 = \|0\| = \|x^*x\| = \|x\|^2$ it follows that $x = 0$ and thus $aa^* = a$.

iii) ⇔ iv) The same as i) ⇔ ii).

□

With this lemma, we can show another defining property of isometries, following [Mur90, Theorem 2.3.3]

Corollary 2.9.33.
Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra and $u \in A$ be a partial isometry. Then, and only then, u is an isometry on $\mathrm{Ker}(u)^\perp$.

Proof 2.9.34.

⇒:

First we want to show that $\mathrm{Ker}(u)^\perp \subset u^*u\mathcal{H}$. A result from functional analysis is, that $\mathrm{Ker}(u)^\perp = \overline{\mathrm{im}(u^*)} = \overline{u^*\mathcal{H}}$, such that it is enough to show that $\overline{u^*\mathcal{H}}$. Since $u^* = u^*uu^*$ by lemma 2.9.31, it holds that:

$$u^*\psi = u^*uu^*\psi = u^*u\widetilde{\psi} \in u^*u\mathcal{H} \ .$$

For the closure, it is enough to take a net and using continuity.

Hence, let $\psi \in \mathrm{Ker}(u)^\perp$, then, because u^*u is a projection, there is a $\phi \in \mathcal{H}$, such that

$$u^*u\psi = u^*uu^*u\phi = (u^*u)^2\phi = u^*u\phi = \psi \ .$$

It follows that:

$$\|u\psi\|^2 = \langle u\psi \mid u\psi\rangle = \langle u^*u\psi \mid \psi\rangle = \langle \psi \mid \psi\rangle = \|\psi\|^2 \ .$$

⇐:

Let p be a projection of $\mathcal{H}$ onto $\mathrm{Ker}(u)^\perp$ and $\psi \in \mathrm{Ker}(u)^\perp$. Then

$$\langle u^*ux \mid x\rangle = \|u\psi\|^2 = \|\psi\| = \langle \psi \mid \psi\rangle = \langle p\psi \mid \psi\rangle \ .$$

On the other hand, if $\psi \in \mathrm{Ker}(u)$, then

$$\langle u^*ux \mid x\rangle = \|u\psi\|^2 0 = \langle p\psi \mid \psi\rangle \ ,$$

such that $\langle u^*ux \mid x\rangle\langle p\psi \mid \psi\rangle$ for all $\psi \in \mathcal{H}$, showing that $p = u^*u$.

□

Corollary 2.9.35.
Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra with SOT closure M. Then M is a von Neumann algebra on $\overline{A\mathcal{H}} \subset \mathcal{H}$. If A acts non-degenerately on $\mathcal{H}$, then it also holds that $M = A^{cc}$.

Proof 2.9.36.
Let (u_λ) be an approximate unit of A. By lemma 2.9.6 (u_λ) converges in SOT against a positive $e \in M$. Hence, in SOT, $u_\lambda a \to ea$ for all $a \in A$. But since (u_λ) is an approximate unit, $u_\lambda a \to a$ in the norm topology. Thus, $e \in M$ is a unit for M. Then, for all $\psi \in A\mathcal{H}$, it holds that $(\mathbb{1} - e)\psi = 0$. Also, for every net in $\psi_\lambda \in \mathcal{H}$ it holds that:

$$\lim_{\lambda\in\Lambda}(\mathbb{1} - e)\psi_\lambda = \lim_{\lambda\in\Lambda} 0 = 0 \ ,$$

showing that $e = \mathbb{1}_{\overline{A\mathcal{H}}} \in M$. Let $(a_\lambda), (b_\lambda)$ be nets, SOT converging against $a, b \in M$. Because of

$$\|(ta + (1-t)b - (ta_\lambda + (1-t)b_\lambda))\psi\| \leq t\|(a - a_\lambda)\psi\| + (1-t)\|(b - b_\lambda)\psi\|$$

$ta_\lambda + (1-t)b_\lambda$ SOT converges against $ta + (1+t)b$, showing that M is convex. By corollary 2.9.12 M is WOT closed, and thus $M^* = M$. Hence M is a C^*-sub algebra of $\mathcal{L}(\overline{A\mathcal{H}})$, containing $\mathbb{1}_{\overline{A\mathcal{H}}}$, that is SOT closed, i.e. a von Neumann algebra.

If A acts non-degenerately on $\mathcal{H}$, then $\overline{A\mathcal{H}} = \mathcal{H}$ and thus $e = \mathbb{1}_{\overline{A\mathcal{H}}} = \mathbb{1}_{\mathcal{H}}$. Hence M is a von Neumann algebra on $\mathcal{H}$. From $A \subset M$ it follows that $A^{cc} \subset M^{cc} = M$, using that M is a von Neumann algebra. Furthermore, since A is convex, its WOT closure is M and by applying corollary 2.9.17 twice, A^{cc} is WOT dense in $M^{cc} = M$. But by applying corollary 2.9.15 twice, we see that A^{cc} is already WOT closed. Hence $A^{cc} = M^{cc} = M$. □

Theorem 2.9.37 (**Polar decomposition**)**.**
Let $M \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $x \in M$. Then there is a unique partial isometry $u \in M$, such that u^*u is the projection onto $\overline{|x|\mathcal{H}}$ and $x = u|x|$.

Proof 2.9.38.
Let p be the projection onto $\overline{|x|\mathcal{H}}$ and define u on $|x|\mathcal{H}$ by

$$u(|x|\psi) := x\psi \ , \qquad \forall\ \psi \in \mathcal{H} \ .$$

Since x is fixed, this operator is well defined. As positive element, $|x|$ is self adjoint, and thus

$$\||x|\psi\|^2 = \langle\psi \mid |x|^2\psi\rangle = \langle\psi \mid x^*x\psi\rangle = \|x\psi\|^2 \ ,$$

showing that u is an isometry of $|x|\mathcal{H}$. Hence u extends to an isometry $u\colon p(\mathcal{H}) = \overline{|x|\mathcal{H}} \to \overline{x\mathcal{H}} \subset \mathcal{H}$. Furthermore, this shows that $\mathrm{Ker}(x) = \mathrm{Ker}(|x|)$. By construction (and self adjointness $|x|^* = |x|$) it holds that

$$\mathrm{Ker}(x) = \mathrm{Ker}(|x|) = \left(\overline{|x|\mathcal{H}}\right)^{\perp} = (\mathbb{1} - p)\mathcal{H} \ .$$

Thus $u \equiv 0$ on $(\mathbb{1} - p)\mathcal{H} = \mathrm{Ker}(|x|)$, showing that $\mathrm{Ker}(u) = \mathrm{Ker}(|x|)$. From $\mathrm{Ker}(u)^{\perp} = \mathrm{Ker}(|x|)^{\perp} = \overline{|x|\mathcal{H}}$ and corollary 2.9.33 it follows that u is a partial isometry and $u^*u = p$ is the projection onto $\overline{|x|\mathcal{H}}$. Also, by construction it holds that $u|x| = x$.

Let now $v \in M$, such that $v^*v = p$ and $v|x| = x$. Hence

$$v = vv^*v = vp \qquad \text{and} \qquad u = uu^*u = up$$

and for all $\psi \in \mathcal{H}$:

$$v|x|\psi = x\psi = u|x|\psi \ .$$

Thus $vp = up$, showing $u = v$.

It remains to show, that $u \in M$. Let $A = C^*(x) \subset M$. Hence, it is enough to show that $x \in A^{cc} \subset M^{cc} = M$. Let $y \in A^c$, then it holds that

$$xy(\mathbb{1} - p)\psi = yx(\mathbb{1} - p)\psi = 0 \quad \Rightarrow \quad y(\mathrm{Ker}(x)) \subset \mathrm{Ker}(x) = (\mathbb{1} - p)\mathcal{H} \ .$$

From $\mathrm{Ker}(u) = \mathrm{Ker}(|x|) = \mathrm{Ker}(x)$ it follows that $y(\mathrm{Ker}(u)) \subset \mathrm{Ker}(u)$, i.e. for all $\psi \in \mathcal{H}$

$$uy(\mathbb{1} - p)\psi = yu(\mathbb{1} - p)\psi = 0 \ .$$

On the other hand, $p(\mathcal{H}) = \overline{|x|\mathcal{H}}$ and since $|x| \in A$ because of the functional calculus, for all $\psi \in \mathcal{H}$, using the definition of u it holds that:

$$uy(|x|\psi) = u(|x|y\psi) = xy\psi = yx\psi = yu(|x|\psi) \ .$$

Hence $uyp = yup$ and thus $uy = yu$. □

Lemma 2.9.39.
Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra with SOT closure M. For all $x \in M$, all finite dimensional sub spaces $V \subset \mathcal{H}$, such that the projection p onto V is an element in p and all $\varepsilon > 0$ there is an $a \in A$, such that

$$\|(a - x)|_V\| \leq \varepsilon \qquad \textit{and} \qquad \|a\| \leq \|x|_V\| \ .$$

Proof 2.9.40.
Œ let $\|x|_V\| = 1$ (the case $x|_V = 0$ is trivial). Let $y = xp$, then $\|y\| = 1$. Let $v_1, \ldots, v_n$ be an orthonotmal basis of V. Because of the Kaplansky density theorem 2.9.28, there is an $a \in A$ with $\|a\| \leq 1$, such that

$$\|av_j - xv_j\| = \|av_j - yv_j\| \leq \frac{\varepsilon}{n} \quad \forall j = 1, \ldots, n \ .$$

$$\Rightarrow \qquad (a-y)v_j \leq \varepsilon v_j \qquad \forall\, j=1,\dots,n\ .$$

Then for all $v = \sum_{j=1}^{n} \langle v \mid v_j \rangle v_j \in V$, with $\|v\| \leq 1$, it holds that $|\langle v \mid v_j \rangle| \leq 1$ and thus:

$$\|(a-x)v\| \leq \sum_{j=1}^{n} |\langle v \mid v_j \rangle| \|(a-x)v_j\| \leq \frac{\varepsilon}{n} \sum_{j=1}^{n} \|v_j\| = \frac{\varepsilon}{n} \cdot n\ .$$

This shows that:

$$\|(a-x)|_V\| = \|(a-x)p\| \leq \varepsilon\ .$$

□

Theorem 2.9.41 (**Kadison transitivity theorem**).
Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra with SOT closure M. For all $x \in M$, all finite dimensional sub spaces $V \subset \mathcal{H}$, such that the projection p onto V is an element in p and all $\varepsilon > 0$ there is an $a \in A$, such that

$$a|_V = x|_V \qquad \textit{and} \qquad \|a\| \leq \|y\| + \varepsilon\ .$$

Proof 2.9.42.
By lemma 2.9.39 there is an $a_0 \in A$, such that $\|a_0\| \leq \|x\|$ and $\|(a_0 - x)|_V\| \leq \frac{\varepsilon}{2}$. We are going to show by induction, that for all $n \in \mathbb{N}$ there is an $a_n \in A$, such that $\|a_n\| \leq \frac{\varepsilon}{2^n}$ and

$$\left\| \left(\sum_{k=0}^{n} a_n - x \right) \Bigg|_V \right\| \leq \frac{\varepsilon}{2^{n+1}}\ .$$

Indeed, for $n = 1$, lemma 2.9.39 shows that there is an $a_1 \in A$, such that

$$\|a_1\| \leq \|(x-a)|_V\| \leq \frac{\varepsilon}{2} \qquad \text{and} \qquad \|(x - a_0 - a_1)|_V\| \leq \frac{\varepsilon}{4}\ .$$

Assuming to have proven the statement for $n \in \mathbb{N}$, then lemma 2.9.39 shows the existence of an $a_{n+1} \in A$. such that

$$\|a_{n+1}\| \leq \left\| \left(x - \sum_{k=0}^{n} a_k \right) \Bigg|_V \right\| \leq \frac{\varepsilon}{2^n}$$

$$\text{and} \qquad \left\| \left(\sum_{k=0}^{n} a_n - x \right) \Bigg|_V \right\| \leq \frac{\varepsilon}{2^{n+1}}\ .$$

Hence for $a = \sum_{k=0}^{\infty} a_n \in A$ it follows that $a|_V = x|_V$ and

$$\|a\| \leq \|x \ + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}| = \|x\| + \varepsilon\ .$$

□

2.10 Irreducible $*$-representations

Let K be a convex set. An element $x \in K$ is called extreme point, if it is only an ending point of straight lines in K. That is, from $x = ty_1 + (1-t)y_2$ with $t \in [0,1]$ and $y_1, y_2 \in K$ it follows that $x = y_1$ or $x = y_2$.

Definition 2.10.1.
A state $\phi \in S(A)$ is called **pure state**, if it is an extreme point of $Q(A)$, where $Q(A)$ is the set from lemma 2.8.15.

The following theorems from [Mur90, theorems 5.1.2, 5.1.4 and 5.1.7] will be needed in the proof of theorem 2.10.11:

Theorem 2.10.2.
Let A be a C^*-algebra, ϕ be a state and φ be a positive linear functional, such that $\varphi \leq \phi$, i.e. $\phi - \varphi$ is positive. Then there is a unique operator $V \in \pi_\phi(A)^c$, such that

$$\varphi(a) = \langle \pi_\phi(a) V \psi_\phi \mid \psi_\phi \rangle$$

for all $a \in A$ and $0 \leq V \leq \mathbb{1}$.

Proof 2.10.3.
Let σ be the hermitian scalar product from corollary 2.8.6 defined by $\sigma([a],[b]) := \varphi(b^*a)$. The corollary states, that this is a well defined hermitian scalar product on $\mathcal{H}_\phi$. Using the Cauchy-Schwarz inequality (theorem 2.7.4), we see that:

$$\begin{aligned} |\sigma([a],[b])| = |\varphi(b^*a)| &\leq \sqrt{\varphi(b^*b)}\sqrt{\varphi(a^*a)} \\ &\leq \sqrt{\phi(b^*b)}\sqrt{\phi(a^*a)} = \|[b]\|_\phi \cdot \|[a]\|_\phi \ , \end{aligned}$$

showing that $\|\sigma\| \leq 1$. Then there is an operator $V = vu^*\mathcal{L}(\mathcal{H}_\phi)$, where $v, u \in \mathcal{L}(\mathcal{H}_\phi)$, with $\|V\| \leq 1$ such that

$$\langle V\psi \mid \psi' \rangle = \langle v\psi \mid u\psi' \rangle = \sigma(\psi, \psi') \qquad \forall\ \psi, \psi' \in \mathcal{H}_\phi \ .$$

From $\sigma(\psi,\psi) = \langle V\psi \mid \psi \rangle$ it also follows that $V \geq 0$. It also holds that:

$$\varphi(b^*a) = \sigma([a],[b]) = \langle v[a],[b] \rangle = \langle v\pi_\phi(a)\psi_\phi, \pi_\phi(b)\psi_\phi \rangle$$

where we used $\pi(a)[b] = [ab]$ and theorem 2.8.8 for $\phi(b^*Va)$ in the last step. Let $a, b, c \in A$, then:

$$\begin{aligned} \langle \phi(a)V[b] \mid [c] \rangle &= \langle V[b] \mid [a^*c] \rangle = \varphi(c^*ab) \\ &= \langle V[ab] \mid [c] \rangle = \langle V\pi_\phi(a)[b] \mid [c] \rangle \ . \end{aligned}$$

$$\Rightarrow \qquad \pi_\phi(a)V = V\pi_\phi(a) \quad \forall\ a \in A \qquad \Leftrightarrow \qquad V \in \pi_\phi(A)^c \ .$$

Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of A, then:

$$\varphi(u_\lambda a) = \langle V[a] \mid [u_\lambda] \rangle = \langle V\pi_\phi(a)\psi_\phi \mid \pi_\phi(u_\lambda)\psi_\phi \rangle = \langle V\pi_\phi(u_\lambda a)\psi_\phi \mid \psi_\phi \rangle \ .$$

In the limit, this shows that

$$\varphi(a) = \langle V\pi_\phi(a)\psi_\phi \mid \psi_\phi\rangle = \langle \pi_\phi(a)V\psi_\phi \mid \psi_\phi\rangle \ .$$

For uniqueness, assume $U \in \pi_\phi(A)^c$ also satisfies
$\varphi(a) = \langle \pi_\phi(a)U\psi_\phi \mid \psi_\phi\rangle$. Then

$$\begin{aligned}\langle U[a],[b]\rangle &= \langle U\pi_\phi(b^*a)\psi_\phi \mid \psi_\phi\rangle = \varphi(b^*a)\\ &= \langle U\pi_\phi(b^*a)\psi_\phi \mid \psi_\phi\rangle = \langle U[a],[b]\rangle\end{aligned}$$

for all $a, b \in A$ and thus $U = V$. □

Corollary 2.10.4.
Choosing $U = V^*$ it follows that

$$\varphi(a) = \langle \psi_\phi \mid \pi_\phi(a)V\psi_\phi\rangle \ .$$

Theorem 2.10.5.
*Let (H_j, π_j) for $j = 1, 2$ be two representations of A with cyclical vectors $\psi_j \in \mathcal{H}_j$. Then there is a unitary operator $U: \mathcal{H}_1 \to \mathcal{H}_2$, such that $\psi_2 = U\psi_1$ and $\pi_2(a) = U\pi_1(a)U^*$ for all $a \in A$, i.e. the representations are unitarily **equivalent** if and only if*

$$\langle \pi_1(a)\psi_1 \mid \psi_1\rangle = \langle \pi_2(a)\psi_2 \mid \psi_2\rangle \qquad \forall\ a \in A\ .$$

Proof 2.10.6.
Assume that $\langle \pi_1(a)\psi_1 \mid \psi_1\rangle = \langle \pi_2(a)\psi_2 \mid \psi_2\rangle$ for all $a \in A$. Define a linear operator $U_0: \pi_1(A)\psi_1 \to \mathcal{H}_2$ by

$$U_0(\pi_1(a)\psi_1) = \pi_2(a)\psi_2\ .$$

From

$$\|\pi_2(a)\psi_2\|^2 = \langle \pi_2(a^*a)\psi_2 \mid \psi_2\rangle = \langle \pi_1(a^*a)\psi_1 \mid \psi_1\rangle = \|\pi_1(a)\psi_1\|^2\ ,$$

showing that U_0 is a well defined isometry. Let $U: \mathcal{H}_1 \to \mathcal{H}_2$ be the extension of U_0. Since $U(\mathcal{H}_1) = \overline{\pi_2(A)\psi_2} = \mathcal{H}_2$, it is an unitary operator. For $a, b \in A$ it holds that

$$\begin{aligned}U\pi_1(a)\pi_1(b)\psi_1 &= \pi_2(ab)\psi_2 = \pi_2(a)U\pi_1(b)\psi\\ \Rightarrow \quad U\pi_1(a) &= \pi_2(a)U \quad \forall\ a \in A\ .\end{aligned}$$

Finally:

$$\pi_2(a)U\psi_1 = U\pi_1(a)\psi_1 = \pi_2(a)\psi_2 \quad \Rightarrow \quad \pi_2(a)(U\psi_1 - \psi_2) = 0\ .$$

As representation with cyclical vector, the representations are non-degenerate, and thus $U\psi_1 = \psi_2$.

The opposite direction is a direct calculation:

$$\langle \pi_2(a)\psi_2 \mid \psi_2\rangle = \langle U\pi_1(a)U^*\psi_2 \mid \psi_2\rangle\langle \pi_1(a)U^{-1}\psi_2 \mid U^-1\psi_2\rangle = \langle \pi(a)\psi_1 \mid \psi_1\rangle$$

for all $a \in A$ □

Theorem 2.10.7.
Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra and $\psi \in \mathcal{H}$ be cyclical with $\|\psi\| = 1$. Then the function

$$\phi \colon A \longrightarrow \mathbb{C} \ , \qquad a \longmapsto \langle \psi \mid \pi(a)\psi \rangle$$

is a state of A, and $(\mathcal{H}, \pi)$ is unitarily equivalent[5] *to $(\mathcal{H}_\phi, \pi_\phi)$.*

Proof 2.10.8.
Since ψ is cyclical, the representation is non-degenerate and by corollary 2.8.2 it holds that for an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ the net $(\pi(u_\lambda))_\lambda$ is SOT convergent against $\mathbb{1}_{\mathcal{H}}$. By example 2.8.4 ϕ is a positive linear functional and theorem 2.7.10 applies, such that

$$\|\phi\| = \lim_{\lambda\in\Lambda} \phi(u_\lambda) = \lim_{\lambda\in\Lambda} \langle \psi \mid \pi(u_\lambda)\psi \rangle = \langle \psi \mid \psi \rangle = \|\psi\| = 1 \ .$$

Hence ϕ is a state of A. Because of theorem 2.8.8 it follows that for all $a \in A$

$$\langle \psi_\phi \mid \pi_\phi(a)\psi_\phi \rangle_\phi = \phi(a) = \langle \psi \mid \pi(a)\psi \rangle \ ,$$

which is the unitarily equivalence by theorem 2.10.5. □

Lemma 2.10.9.
Let $M \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then M is the smallest von Neumann algebra that contains all projections $p = p^* = p^2 \in M$.

Proof 2.10.10.
Let $a = a^* \in M$. By the spectral theorem for self adoint operators a is in the SOT closure of the convex hull of its spectral projections, which are SOT limits of polynomials of a. □

Theorem 2.10.11.
Let $(\mathcal{H}, \pi)$ be a non-trivial $*$-representation of A, then the following claims are equivalent:

i) The representation π is irreducible, i.e. there is no subspace $\emptyset \neq V \subsetneq \mathcal{H}$, that is invariant under the action of $\pi(A)$.

ii) The representation π is topological irreducible, i.e. there is no closed subspace $\emptyset \neq V \subsetneq \mathcal{H}$, that is invariant under the action of $\pi(A)$.

[5] By unitarity it follows that u is invertible and thus an isomorphism, and by unitarity it is also an isometry.

iii) *The only projection* $p = p^* = p^2 \in \pi(A)^c$ *are* 0 *and* $\mathbb{1}_{\mathcal{H}}$.

iv) *It holds that* $\pi(A)^c = \mathbb{C}\mathbb{1}_{\mathcal{H}}$.

v) *The set* $\pi(A)$ *is SOT dense in* $\mathcal{L}(\mathcal{H})$.

vi) *For all* $\xi, \psi \in \mathcal{H}$ *with* $\psi \neq 0$ *there is an* $a \in A$*, such that* $\pi(a)\psi = \xi$.

vii) *Every vector* $0 \neq \psi \in \mathcal{H}$ *is cyclical.*

viii) *There is a pure state* $\phi \in S(A)$ *and an isometrical isomorphism* $u\colon \mathcal{H} \to \mathcal{H}_\phi$*, such that* $u\pi(a) = \pi_\phi(a)u$ *for all* $a \in A$.

Proof 2.10.12.

i) ⇒ ii): This is obvious.

ii) ⇒ iii): Let $p = p^* = p^2 \in \pi(A)^c$, then $p(\mathcal{H})$ is closed and $\pi(A)$-invariant, for the same reason as $e\mathcal{H}$ in the proof of corollary 2.9.35. Hence $p = 0$ or $p = \mathbb{1}_{\mathcal{H}}$.

iii) ⇒ iv): From lemma 2.10.9 it follows that $\pi(A)^c$ is generated by $p = 0$ and $p = \mathbb{1}_{\mathcal{H}}$.

iv) ⇒ v): Assume that π is degenerate, then $(\pi(A)\mathcal{H})^\perp \neq 0$. Let $p = p^2 = p^*$ be the projection onto $(\pi(A)\mathcal{H})^\perp$, so $p \neq 0$. And since $\pi \neq 0$, $p \neq \mathbb{1}$. To see that $p \in \pi(A)^c x$ we observe that $p\pi(a)\psi = 0$. On the other hand:

$$\langle \psi' \mid \pi(a)p\psi \rangle = \langle \pi(a^*)\psi' \mid p\psi \rangle = 0 \quad \Rightarrow \quad \pi(a)p = 0$$

Hence $\pi(a)p = p\pi(a) = 0$. But this contradicts $\pi(A)^c = \mathbb{C}\mathbb{1}_{\mathcal{H}}$, so π is non-degenerate.

Let M denote the SOT closure of $\pi(A)$. Since $\pi(A)$ is a C^*-sub algebra of $\mathcal{L}(\mathcal{H})$ that acts non-degenerately on $\mathcal{H}$, it holds that $M = \pi(A)^{cc}$, by corollary 2.9.35. But since $\pi(A)^c = \pi(A)^c = \mathbb{C}\mathbb{1}_{\mathcal{H}}$ it follows that $\pi(A)^{cc} = \mathcal{L}(\mathcal{H}) = M$. Hence $\pi(A)$ is SOT dense in $\mathcal{L}(\mathcal{H})$.

v) ⇒ vi): Let $T \in \mathcal{L}(\mathcal{H})$, such that $T\psi = \xi$. Using theorem 2.9.41 for the finite dimensional subspace $V := \langle \psi, \xi \rangle_{\mathbb{C}} \subset \mathcal{H}$ shows that there is an $\pi(a) \in \pi(A)$ with $\pi(a)|_V = T_V$. Hence $\pi(a)\psi = \xi$.

vi) ⇒ i): Let $\psi \neq 0$. The smallest invariant subspace, that contains ψ, also contains $\pi(A)\psi = \mathcal{H}$.

vi) ⇒ vii): By definition.

vii) ⇒ iii): Let $0 \neq p = p^* = p^2 \in \pi(A)^c$ and $0 \neq \psi \in p(\mathcal{H})$. Then $\pi(A)\psi \subset p(\mathcal{H})$. But by assumption, $pi(A)\psi$ is dense in $\mathcal{H}$, such that $p = \mathbb{1}_{\mathcal{H}}$.

iv) ⇒ viii): By vii), ever non-zero vector is cyclical. Because of theorem 2.10.7, we can assume Œ, that $(\mathcal{H},\pi) = (\mathcal{H}_\phi,\pi_\phi)$ for a state $\phi \in S(A)$.

Let $0 \le t \le 1$ and $\phi = t\phi_1 + (1-t)\phi_2$ for $\phi_1,\phi_2 \in S$. Then it holds that $0 \le t\phi_1 \le \phi$, such that by theorem 2.10.2, there is an $x \in \pi(A)^c$ with $0 \le x$ with

$$t\phi_1(a) = \langle \psi_\phi \mid \pi(a)x\psi_\phi\rangle \qquad \forall\ a \in A\ .$$

Then, because of iv), $x = \lambda \mathbb{1}_{\mathcal{H}}$ for $\lambda \ge 0$. It follows that (theorem 2.8.8):

$$t\phi(a) = \langle \psi_\phi \mid \lambda\pi_\phi(a)\psi_\phi\rangle = \lambda\phi(a) \qquad \forall\ a \in A\ .$$

Thus $t = \|t\phi_1\| = \|\lambda\phi\| = \lambda$, so either $t = 0$ or $\phi = \phi_1$. Hence ϕ is a pure state.

viii) ⇒ iii): Let ϕ be a pure state and $(\mathcal{H},\pi) = (\mathcal{H}_\phi,\pi_\phi)$ the associated GNS $*$-representation with cyclical vector $\psi = \psi_\phi$. Œ we can set $\|\psi\| = 1$. Assume, that there is a $p = p^* = p^2 \in \pi(A)^c$ with $p \ne 0$ and $p \ne \mathbb{1}_{\mathcal{H}}$. Let ϕ_1,ϕ_2 defined by

$$\phi_1(a) \coloneqq \langle \psi \mid \pi(a)p\psi\rangle \quad \text{and} \quad \phi_2(a) \coloneqq \langle \psi \mid \pi(a)(\mathbb{1}_{\mathcal{H}} - p)\psi\rangle$$

for all $a \in A$. Hence, $\phi = \phi_1 + \phi_2$. Since p and $(\mathbb{1}_{\mathcal{H}} - p)$ are projections and by example 2.8.4 it holds that

$$\phi_1(a^*a) = \|\pi(a)p\psi\|^2 \qquad \text{and} \qquad \phi_2(a^*a) = \|\pi(a)(\mathbb{1}_{\mathcal{H}} - p)\psi\|\ ,$$

showing that ϕ_1 and ϕ_2 are positive. If $p(\psi) = 0$, then $p(\pi(A)\psi) = 0$ and thus $p = 0$, which is a contradiction to the assumption. Hence $p(\psi) \ne 0$, and so $(\mathbb{1}_{\mathcal{H}} - p)(\psi) \ne 0$. For the operator norm it holds that (using corollary 2.8.2 and that the scalar product is continuous):

$$0 < \|p\psi\|^2 = \|\phi_1\| \qquad \text{and} \qquad 0 < \|((\mathbb{1}_{\mathcal{H}}) - p)\pi(a)\psi\|^2 = \|\phi_2\|\ .$$

But then

$$\begin{aligned}\|\phi_1\| + \|\phi_2\| &= \|p\psi\|^2 + \|((\mathbb{1}_{\mathcal{H}}) - p)\psi\|^2 \\ &= \langle p\psi \mid p\psi\rangle + \langle((\mathbb{1}_{\mathcal{H}}) - p)\psi \mid ((\mathbb{1}_{\mathcal{H}}) - p)\psi\rangle \\ &= \langle \psi \mid \psi\rangle = \|\psi\|^2 = 1\ .\end{aligned}$$

Since $\frac{\phi_i}{\|\phi_i\|} \in S(A)$, this shows that

$$\phi = \phi_1 + \phi_2 = \|\phi_1\|\frac{\phi_1}{\|\phi_1\|} + \|\phi_2\|\frac{\phi_2}{\|\phi_2\|}$$

is a convex sum. But since ϕ is pure, it follows Œ that $\phi_1 = \|\phi_1\|\phi$. Then for all $a,b \in A$:

$$\begin{aligned}\langle [a] \mid p[b]\rangle &= \langle \pi(b)\psi \mid p\pi(a)\psi\rangle = \langle \psi \mid \pi(a^*b)p\psi\rangle = \phi_1(a^*b) \\ &= \|\phi_1\|\phi(a^*b) = \|\phi\|\langle [a] \mid [b]\rangle\ ,\end{aligned}$$

showing that $p = \|\phi_1\|\mathbb{1}_{\mathcal{H}}$. But since p is a projection, it holds that $p^2 = p$, so $\|\phi_1\| = 1$. This shows that $p = \mathbb{1}_{\mathcal{H}}$, which is a contradiction. Thus the claim follows.

□

Definition 2.10.13.
Let $(\mathcal{H}, \pi)$ be a non-trivial $*$-representation. It is called **irreducible**, if it satisfies the equivalent properties form theorem 2.10.11.

3

K-theory

The K-theory of C^*-algebras introduces introduces a sequence of functors, which can be arranged in a long exact sequence. To formulate K-theory, as the usage of "functor" in the previous sentence already suggests, concepts from category theory are useful. The concepts necessary for our purposes are briefly introduced in the first section. The main result of K-theory is the Bott-periodicity, discussed at the end of the chapter.

3.1 Category theory: inductive limit

In this chapter, the concept of inductive limit will be needed. Here, we will consider the general construction for sets with algebraic structures. Although we do not intend to develop category theory here, it presents itself as helpful to use the terminology:

3.1.1 Categorical language

Definition 3.1.1.
A **category** $\mathcal{C}$ is a collection of **objects**, denoted by $\mathrm{Ob}(\mathcal{C})$ and a set $\mathrm{mor}(A, B)$ of maps between the objects $A, B \in \mathrm{Ob}(\mathcal{C})$, called **morphisms**, such that the following properties hold:

i) There is an identity morphism $\mathrm{Id}_A \in \mathrm{mor}(A, A)$.

ii) There is a notion of composition: $\circ\colon \mathrm{mor}(B, C) \times \mathrm{mor}(A, B) \longrightarrow \mathrm{mor}(A, C)$, that has to be associative.

iii) The identity acts as its name suggests, under these compositions.

The the objects form a set, the category is called **small**.

On example for a category is the category of C^*-algebras, where the morphisms are defined to be the $*$-morphisms (hence the name).

Definition 3.1.2.
A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a map between categories $\mathcal{C}$ and $\mathcal{D}$, such that

i) $F\colon \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$, $A \longmapsto F(A)$,

ii) $F\colon \mathrm{mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{mor}_{\mathcal{D}}(F(A), F(B))$, $f \longmapsto F(f)$

iii) $F(\mathrm{Id}_A) = F_{\mathrm{Id}_A}$ and $F(g \circ f) = F(g) \circ F(f)$.

Functors can be summarized by the commutativity of the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ F(A) & \xrightarrow[F(f)]{\qquad} & F(B) \end{array}$$

The type of functors defined above is called **covariant**. A **contravariant** functor reverses the direction of $F(f)$, i.e. $F(f)\colon F(B) \to F(A)$.

> **Definition 3.1.3.**
> Let, $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** of functors $\alpha\colon F \to G$ consists of maps $\alpha_A\colon F(A) \to G(A)$ for all objects $A \in \mathrm{Ob}(\mathcal{C})$, such that for all morphisms $f \in \mathrm{mor}_{\mathcal{C}}(A, B)$ the following diagram commutes:
>
> $$\begin{array}{ccc} F(A) & \xrightarrow{\ F(f)\ } & F(B) \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{\qquad} & G(B) \end{array}$$

For the next terminology, we recall the concept of short exact sequences. The sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

is called **short exact sequence** if

$$0 = \mathrm{Ker}(f)\ , \quad \mathrm{Im}(f) = \mathrm{Ker}(g)\ , \quad \mathrm{Im}(g) = C\ .$$

> **Definition 3.1.4.**
> A functor is called **exact**, if it preserves exact sequences.

3.1.2 Direct limits

We have already met the notion of directed sets (definition 2.5.1). We define:

> **Definition 3.1.5.**
> Let $\mathcal{C}$ be a category. A **direct system** in $\mathcal{C}$ is the tuple $(\{C_\lambda\}_{\lambda\in\Lambda}, \phi_{\lambda\mu})$, where Λ is a directed set, $\{C_\lambda\}_\lambda$ a family of objects and $\phi_{\lambda\mu} \in \mathrm{mor}(C_\lambda, C_\mu)$ are morphisms, such that the following properties hold:
>
> i) $\phi_{\lambda\lambda} = \mathrm{Id}_{C_\lambda}$ for all $\lambda \in \Lambda$.
>
> ii) $\phi_{\mu\nu} \circ \phi_{\lambda\mu} = \phi_{\lambda\nu}$ for all $\lambda \le \mu \le \nu$.

By definition of morphisms, $\phi_{\mu\nu} \circ \phi_{\lambda\mu} \in \mathrm{mor}(C_\lambda, C_\nu)$. But it need not be the special morphism $\phi_{\lambda\nu}$.

Example 3.1.6.
A special direct system, used in this chapter is a normed direct system. Let $\mathcal{A}$ be a category of normed algebras (or banach algebras, etc.) and $(\{A_\lambda\}_{\lambda\in\Lambda}, \phi_{\lambda\mu})$ a direct system. If

$$\|a\|' := \limsup_{\mu\in\Lambda} \|\phi_{\lambda\mu}(a)\|_{A_\mu} < \infty \qquad \forall\ a \in A_\lambda\ ,$$

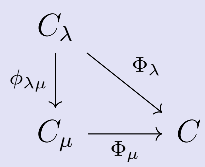
then the direct system is called **normed direct system**. Also, $\|\cdot\|'$ is a semi norm.
The limit superior for a net $(x_\mu)_{\mu\in\Lambda}$ in $\mathbb{R}$ is defined as usual:

$$\limsup_{\mu\in\Lambda} x_\mu = \lim_{\mu\in\Lambda} \sup_{\nu\geq\mu} x_\nu = \inf_{\mu\in\Lambda} \sup_{\nu\geq\mu} x_\nu\ .$$

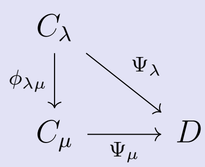
Since $x_\mu := \|\phi_{\lambda\mu}(a)\|_{A_\mu}$ is a net in $\mathbb{R}$, the limit superior is well defined in the definition of normed direct systems.

Definition 3.1.7.
Let $(\{C_\lambda\}_{\lambda\in\Lambda}, \phi_{\lambda\mu})$ be a direct system of a category $\mathcal{C}$. An object C is called the **direct limit** $C \equiv \varinjlim C_\lambda$ of the direct set, if there is a morphism $\Phi_\lambda \colon C_\lambda \to C$, for all $\lambda \in \Lambda$, such that the following properties are satisfied:

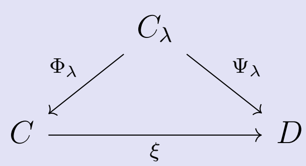
i) For all $\lambda \leq \mu$, the following diagram commutes:



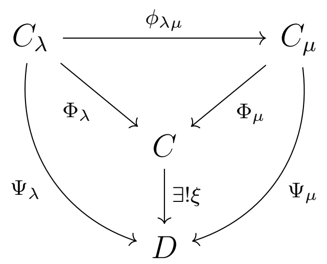
ii) Let $D \in \mathrm{Ob}(\mathcal{C})$ and $\Psi_\lambda \colon C_\lambda \to D$, such that also the diagram



commutes. Then there is a unique morphism $\xi \colon C \to D$, such that the following diagram commutes for all $\lambda \in \Lambda$:



The conditions of the direct limit can be put in a single diagram:



Theorem 3.1.8.
Let (C, ϕ_μ) and (C', ϕ'_μ) be direct limits of the inductive system $(\{C_\lambda\}_{\lambda\in\Lambda}, \phi_{\lambda\mu})$, then there is a unique isomorphism $\xi\colon C \to C'$.

Proof 3.1.9.
By definition of direct limits, there are unique morphisms

$$\begin{aligned} \xi\colon C &\longrightarrow C' \quad \text{such that:} \quad \xi\circ\phi_\mu = \phi'_\mu \\ \xi'\colon C' &\longrightarrow C \quad \text{such that:} \quad \xi'\circ\phi'_\mu = \phi_\mu \ . \end{aligned}$$

We calculate that:

$$\xi\circ\xi'\circ\phi'_\mu = \xi\circ\phi_\mu = \phi'_\mu = \mathrm{Id}_{A'}\circ\phi'_\mu \quad\Rightarrow\quad \xi\circ\xi' = \mathrm{Id}_{A'}$$

and in the same way, that $\xi'\circ\xi = \mathrm{Id}_A$, which shows that ξ is an isomorphism. □

The existence of direct limits is not guaranteed in general. However, in case of algebras with direct set $\mathbb{N}$, it can be constructed explicitly.

Theorem 3.1.10.
Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of algebras, such that (A_i, ϕ_{ij}) is a direct system of algebras, then, if the direct limit exists:

$$\varinjlim A \cong \textstyle\bigsqcup_i A_i/\sim \ ,$$

with the equivalence relation $x_i \in A_i, x_j \in A_j$, $x_i \sim x_j$, if and only if there is a $k \in \mathbb{N}$, such that $i \le k$, $j \le k$ and $\phi_{ik}(x_i) = \phi_{jk}(x_j)$.

Proof 3.1.11.

- First we show, that $\sim$ is indeed an equivalence relation.

  Reflexivity: Follows from the property that $\phi_{ii} = \mathrm{Id}_{A_i}$.

  Symmetry: Follows from the symmetry from $=$.

  Transitivity: Let $i, j \le k$, such that $\phi_{ik}(x_i) = \phi_{jk}(x_j)$ and let $k \le \ell$. Multiplying from left with $\phi_{k\ell}$, using the property of direct systems we find:

$$\phi_{k\ell}\circ\phi_{ik} = \phi_{i\ell} \qquad \text{and} \qquad \phi_{k\ell}\circ\phi_{jk} = \phi_{j\ell}$$

$$\Rightarrow \qquad (\phi_{k\ell} \circ \phi_{ik})(x_i) = \phi_{i\ell}(x_i) = \phi_{j\ell}(x_j) = (\phi_{k\ell} \circ \phi_{jk})(x_j) \ .$$

From this general property, transitivity follows.

- For the equivalence class of $x_i \in A_i$ we write $[x_i, i]$. The algebra structure of $\bigsqcup_i A_i/\sim$ is defined by:

$$[x_i, i] + [x_j, j] = [\phi_{ik}(x_i) + \phi_{jk}(x_j), k]$$

$$\text{and} \quad [x_i, i][x_j, j] = [\phi_{ik}(x_i)\phi_{jk}(x_j), k]$$

for any $k \geq i, j$. For short we write $\diamond$ for any of the operations here. Let $\ell \geq i, j$ and $m \geq k, \ell$, then

$$\phi_{km}(\phi_{ik}(x_i) \diamond \phi_{jk}(x_j)) = \phi_{im}(x_i) \diamond \phi_{jm}(x_j) = \phi_{\ell m}(\phi_{i\ell}(x_i) \diamond \phi_{j\ell}(x_j))$$

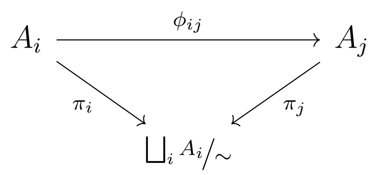
Hence $\phi_{ik}(x_i) \diamond \phi_{jk}(x_j) \sim \phi_{i\ell}(x_i) \diamond \phi_{j\ell}(x_j)$. In the same way, using a larger index and the property of direct systems, one can show that the algebraic structure does not depend on the representative.

- Also we need to show that $\bigsqcup_i A_i/\sim$ has the same mapping property as the direct limit. Let $x_i \in A_i$, then $\phi_{ij}(x_i) \in A_j$. By the property of direct systems, choose $k \geq i, j$:

$$\phi_{ik}(x_i) = (\phi_{jk} \circ \phi_{ij})(x_i) \quad \Leftrightarrow \quad x_i \sim \phi_{ij}(x_i)$$

$$\Leftrightarrow \quad [x_i, i] = [\phi_{ij}(x_i), j] \ .$$

Let $\pi_i \colon A_i \to \bigsqcup_i A_i/\sim, \ x_i \mapsto [x_i, i]$, then by the equality $[x_i, i] = [\phi_{ij}(x_i), j]$ the following diagram commutes:



which is the desired mapping property.

- The π_i are morphisms, because of $[x_i, i] = [\phi_{ij}(x_i), j]$:

$$\begin{aligned} a\pi_i(x_i) + b\pi_i(y_i) &= [ax_i, i] + [by_i, i] = [\phi_{ik}(ax_i) + \phi_{ik}(by_i), k] \\ &= [\phi_{ik}(ax_i + by_i), k] = [ax_i + by_i, i] \\ &= \pi_i(ax_i + by_i) \ . \end{aligned}$$

The same calculation holds for the algebra product.

- Assume now, that there is $(B, \psi_j \colon A_j \to B)$, such that $\psi_k \circ \phi_{jk} = \psi_j$ for all $j \leq k$. Then, it has to be shown that there is a unique morphism $\xi \colon \bigsqcup_i A_i/\sim \to B$ with $\xi \circ \pi_j = \psi_j$.

Existence: Define the function

$$f\colon \bigsqcup_i A_i \to B\ , \qquad f(x_j) \coloneqq \psi_j(x_j) \quad \text{for } x_j \in A_j\ .$$

Let $x_j \sim x_k$, then by definition of $\sim$,

$$\exists\, \ell \in \mathbb{N}\colon j \le \ell\ ,\ k \le \ell \quad \text{and} \quad \phi_{j\ell}(x_j) = \phi_{k\ell}(x_k)\ .$$

Thus, it holds that

$$\begin{aligned} f(x_j) &= \psi_j(x_j) = (\psi_\ell \circ \phi_{j\ell})(x_j) = \psi_\ell(\phi_{j\ell}(x_j)) = \psi_\ell(\phi_{k\ell}(x_k)) \\ &= (\psi_\ell \circ \phi_{k\ell})(x_k) = \psi_k(x_k) = f(x_k)\ . \end{aligned}$$

This means that f only depends on the class $[x_j, j] \in \bigsqcup_i A_i/\sim$. Hence, we can define $\xi\colon \bigsqcup_i A_i/\sim \to B$ by

$$\xi \circ \pi = f\ , \qquad \text{where} \quad \pi(x_j) = \pi_j(x_j)\ .$$

Then, we find that ξ satisfies:

$$(\xi \circ \pi_j)(x_j) = (\xi \circ \pi)(x_j) = f(x_j) = \psi_j(x_j)$$

$$\Rightarrow \qquad \xi \circ \pi_j = \psi_j$$

Morphism property: Let $x, y \in \bigsqcup_i A_i/\sim$, then because of $\sim$, and the algebra structure, there are $j, k \in \mathbb{N}$, such that $x = [x_j, j]$ and $y = [x_k, k]$.

$$\begin{aligned} \xi(x \diamond y) &= \xi([x_i, i] \diamond [x_j, j]) = \xi([\phi_{ik}(x_i) \diamond \phi_{jk}(x_k), k]) \\ &= \xi(\pi_k(\phi_{ik}(x_i) \diamond \phi_{jk}(x_k))) = (\xi \circ \pi)(\phi_{ik}(x_i) \diamond \phi_{jk}(x_k)) \\ &= f(\phi_{ik}(x_i) \diamond \phi_{jk}(x_k)) = \psi_k(\phi_{ik}(x_i) \diamond \phi_{jk}(x_k)) \\ &= (\psi_k \circ \phi_{ik})(x_i) \diamond (\psi_k \circ \phi_{jk})(x_j) = \psi_i(x_i) \diamond \psi_j(x_j) \\ &= f(x_i) \diamond f(x_j) = (\xi \circ \pi)(x_i) \diamond (\xi \circ \pi)(x_j) \\ &= \xi([x_i, i]) \diamond \xi([x_j, j]) = \xi(x) \diamond \xi(y)\ . \end{aligned}$$

Uniqueness: Assume now, that $\zeta\colon \bigsqcup_i A_i/\sim \to B$ be another morphism, such that $\zeta \circ \pi_j = \psi_j$. Let $x \in \bigsqcup_i A_i/\sim$, i.e. $x = [x_k, k]$ for a $k \in \mathbb{N}$. It follows that:

$$\begin{aligned} \zeta(x) &= \zeta([x_k, k]) = \zeta(\pi_k(x_k)) = \psi_j(x_k) = f(x_k) = \xi(\pi_k(x_k)) \\ &= \xi([x_k, k]) = \xi(x)\ . \end{aligned}$$

$$\Rightarrow \zeta = \xi\ .$$

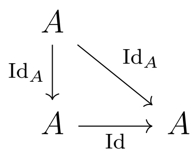
□

Corollary 3.1.12.
Let (A_i, ϕ_{ij}) be a direct system of algebras. For every $a \in \varinjlim A_i$ there is are $j \in \mathbb{N}$

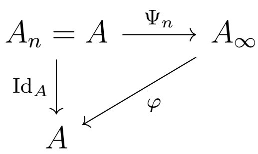
and $x \in A_j$, such that $a = \Phi_j(x)$.

Proof 3.1.13.
This is a direct consequence of the theorem. □

Example 3.1.14.
Let A be an algebra and define $A_n = A$ and $\phi_{mn} = \mathrm{Id}_A$, then (A, Id_A) is a direct system but also satisfies the mapping property (with $\Phi_n = \mathrm{Id}_A$):



Assume now, that (A_∞, Ψ_n) is the direct limit, then there exists a unique morphism $\varphi\colon A_\infty \to A$, such that $\varphi \circ \Psi_n = \mathrm{Id}_A$:



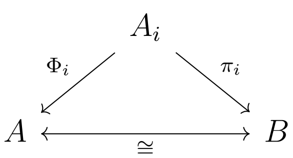
However, then injectivity and surjectivity of φ are immediate. Hence $A_\infty \cong A$.

Corollary 3.1.15.
Let (A_i, ϕ_{ij}) be a direct system of algebras, and let (A, Φ_i) be the direct limit. Then, if the ϕ_{ij} are injective, the Φ_i are also injective.

Proof 3.1.16.
Let (B, π_i) be the direct limit from construction of theorem 3.1.10. Let $x \in A_i$ and $y \in A_j$, such that $\pi_i(x) = \pi_j(y)$. This means, that $x \sim y$. By definition of the equivalence, there is a $\mathbb{N} \ni k \geq i, j$, such that $\phi_{ik}(x) = \phi_{jk}(y)$.

So if $x, x' \in A_i$, such that $\pi_i(x) = \pi_i(x')$, it follows that there is a $\mathbb{N} \ni k \geq i$, such that $\phi_{ik}(x) = \phi_{ik}(x')$. Yet, the map ϕ_{ik} is injective, and thus $x = x'$. Hence π_i is injective.

Since the direct limit is unique up to isomorphy (theorem 3.1.8), it follows that the Φ_i are injective, by the commutativity of the following diagram:



□

In fact, the first k spaces of a direct system over $\mathbb{N}$ do not contribute to the direct limit, i.e. can be neglected as the next lemma shows:

Lemma 3.1.17.
Let (A_n, ϕ_{mn}) be a direct system with limit (A, Φ_n). Then for any $k \in \mathbb{N}$, the direct system $(\{A_n\}_{n>k}, \phi_{mn})$ has the same direct limit.

Proof 3.1.18.
If (A, Φ_n) satisfies the mapping property of (A_n, ϕ_{mn}), it necessarily does so for $(\{A_n\}_{n>k}, \phi_{mn})$.

From corollary 3.1.12 it follows, that for all $a \in A$, there is an $a_m \in A_m$, such that $\Phi_m(a_m) = a$. If now $m \leq k$, consider $n > k$ and define $a_n := \phi_{mn}(a_m)$, then

$$\Phi_n(a_n) = (\Phi_n \circ \phi_{mn})(a_m) = \Phi_m(a_m) = a \ .$$

Hence, the direct limits of (A_n, ϕ_{mn}) and $(\{A_n\}_{n>k}, \phi_{mn})$ contain the same elements.

□

Lemma 3.1.19.
Let (A_n, ϕ_{mn}) and (B_n, ψ_{mn}) be algebraical direct systems with direct limits (A, Φ_n) and (B, Ψ_n). If there are morphisms $\varphi_n \colon A_n \to B_n$, such that the following diagram commutes

$$\begin{CD} A_m @>{\varphi_m}>> B_m \\ @V{\phi_{mn}}VV @VV{\psi_{mn}}V \\ A_n @>>{\varphi_n}> B_n \end{CD}$$

then there is a unique morphism $\varphi \colon A \to B$, such that the following diagram commutes:

$$\begin{CD} A_n @>{\varphi_n}>> B_n \\ @V{\Phi_n}VV @VV{\Psi_n}V \\ A @>>{\exists!\ \varphi}> B \end{CD}$$

Proof 3.1.20.
Let $a \in A$, then there is an $a_n \in A_n$, such that $\Phi_n(a_n) = n$. Define $\varphi(a) = \varphi(\Phi_n(a_n)) = \Psi_n(\varphi_n(a_n))$. Then, the diagram commutes by construction.

Let now $a'_m \in A_m$, such that also $a = \Phi_m(a'_m)$. From the explicit construction of the algebraic direct limit (theorem 3.1.10) it follows, that there is a $\mathbb{N} \ni k \geq m, n$, such that $\phi_{nk}(a_n) = \phi_{mk}(a'_m)$. Thus

$$\begin{aligned}\varphi(\Phi_n(a_n)) &= \varphi((\Phi_k(\phi_{nk}(a_n)))) = \Psi_k(\varphi_k(\phi_{nk}(a_n))) = \Psi_k(\varphi_k(\phi_{mk}(a'_m))) \\ &= \varphi(\Phi_k(\phi_{mk}(a'_m))) = \varphi(\Phi_m(a'_m)) \ .\end{aligned}$$

This shows that φ is well defined.

Let now $f\colon A \to B$ be another morphism, such that the morphism commutes. Then we see that

$$f(a) = f(\Phi_n(a_n)) = \Psi_n(\varphi_n(a_n)) = \varphi(\Phi_n(a_n)) = \varphi(a) \ ,$$

so $f \equiv \varphi$. Hence, φ is unique. □

Lemma 3.1.21.
Let (A_i, ϕ_{ij}) be a normed direct system over $\mathbb{N}$ and $N = \{a \in A \mid \|a\|' = 0\}$. If A is the direct limit of algebras and $\|\cdot\|_\sim$ the quotient norm, then $({}^{A}/{}_{N}, \|\cdot\|_\sim)$ is the direct limit of normed algebras, called ***normed direct limit****.*

Proof 3.1.22.
From corollary 3.1.12 it follows that for all $a \in A$, there is an $i \in \mathbb{N}$ and $x_i \in A_i$, such that $a = \Phi_i(x_i)$. Thus we set $\|a\|' = \|x_i\|'$. With this definition, the projections in the proof of theorem 3.1.10 become continuous. Hence, we can assume the Φ_i to be continuous, i.e. to be morphisms in the category of normed algebras. To show that this is well defined, we assume, that there are $j \in \mathbb{N}$ and $y_j \in A_j$, such that $a = \Phi_j(y_j)$ and Œ $j \geq i$. Since A is the algebraic direct limit, it holds that $\Phi_i = \Phi_j \circ \phi_{ij}$.

$$\Phi_j(y_j) = a = \Phi_i(x_i) = \Phi_j(\phi_{ij}(x_i)) \quad \Rightarrow \quad y_j = \phi_{ij}(x_j)$$

Then, with the definition of $\|\cdot\|'$ and the identity $\phi_{j\ell} \circ \phi_{ij} = \phi_{i\ell}$ for $i \leq j \leq \ell$:

$$\begin{aligned}\|y_j\|' &= \lim_{k\to\infty} \sup_{\ell \geq k} \|\phi_{j\ell}(y_i)\| = \lim_{k\to\infty} \sup_{\ell\geq k} \|\phi_{j\ell}(\phi_{ij}(x_i))\| \\ &= \lim_{k\to\infty} \sup_{\ell\geq k} \|\phi_{i\ell}(x_i)\| = \|x_i\|' \ .\end{aligned}$$

Hence $\|a\|'$ is well defined. Since $\|\cdot\|'$ is a semi norm $\|\cdot\|_\sim$ becomes a semi norm on ${}^{A}/{}_{N}$. It remains to show positive semi definedness:

Assume that $\|a + N\|_\sim = 0$, then

$$0 = \|a + N\|_\sim = \inf_{n\in N} \|a + n\|' \leq \inf_{n\in N} \|a\|' + \|n\|' = \|a\|' \ .$$

But this means, that $a \in N$, i.e. $[a] = [0]$. Conversely, for $a = 0$ it follows that $\|a + N\|_\sim = 0$. Hence $({}^{A}/{}_{N}, \|\cdot\|_\sim)$ is a normed algebra. □

3.2 Local C^*-algebras

Let A be a normed Algebra. The matrix algebra $M_n(A)$ is the set of $n \times n$-matrices with coefficients in A and the usual structure of matrices. Because of the norm of A, there is a norm on A^n, defined by

$$\|x\|_2 := \sqrt{\sum_{j=1}^{n} \|x_j\|^2} \qquad \forall\, x = (x_1, \dots, x_n) \in A^n \ .$$

The operator norm $M_n(A)$ is defined as usual:

$$\|A\| := \sup\{\|Ax\|_2 \mid x \in A^n \ , \ \|x\|_n \leq 1\} \ .$$

Remark 3.2.1.
For the next technical definition, we mention, that there are ways to define the spectrum of non-unital banach algebras. Also, we have considered the functional calculus for C^*-algebras so far. However, it exists for Banach algebras as well, known as the Riesz functional calculus.

Definition 3.2.2.
Let A be a normed algebra. It is called **local Banach algebra**, if the following holds:
Let $n \in \mathbb{N}$ and $\widehat{M_n(A)}$ be the metric completion of the matrix algebra to a Banach algebra. Let f be a holomorphic function defined on an open neighborhood U of the spectrum $\sigma_{\widehat{M_n(A)}}(a)$ and $f(0) = 0$, if $0 \in U$. Then, $f(a) \in M_n(A)$. One says, $M_n(A)$ is **closed under holomorphic functional calculus**.

Note that $M_1(A) = A$. The restricting property of this definition is, that $f(a)$ is only in $\widehat{M_n(A)}$ a priori.

Definition 3.2.3.
Let A be a normed $*$-algebra, that is a local Banach algebra. It is called **local C^*-algebra** if

$$\|a^*a\| = \|a\|^2 \ , \qquad \forall\, a \in A \ .$$

Although more examples can be given, a straightforward example for a local C^*-algebra, by what we have covered so far, is a C^*-algebra.

Lemma 3.2.4.
Let A be a unital local Banach algebra. If $a \in A$ is invertible in $\widehat{A}$, then $a^{-1} \in A$, i.e. $A^\times = A \cap \widehat{A}^\times$. It follows that $\sigma_A(a) = \sigma_{\widehat{A}}(a)$.

Proof 3.2.5.
Consider the function $f(z) = z^{-1}$. Since a is invertible in $\widehat{A}$, it holds that $0 \not\in \sigma_{\widehat{A}}(a)$,

such that f is a proper holomorphic function to be considered. By assumption, A is a local Banach algebra, such that $f(a) = a^{-1} \in A$.

□

Corollary 3.2.6.
Let A be a unital local Banach algebra. Then the invertible elements of A are dense in the invertible elements of $\widehat{A}$.

Proof 3.2.7.
The completion of A is a Banach algebra, and since it is unital by assumption, the set of invertible elements $\mathrm{Inv}(\widehat{A})$ of $\widehat{A}$ is open because of lemma 1.4.7.Also, $\widehat{A}$ is dense in A. Let $a \in \widehat{A}$ be invertible, then, for every open neighborhood $U \subset \mathrm{Inv}(\widehat{A})$ of a, there is an $b \in A$, such that $b \in U$. This means, that b is invertible in $\widehat{A}$, and by the previous lemma, it is invertible in A. □

Lemma 3.2.8.
Let $(\{A_\lambda\}_{\mu\in\Lambda}, \phi_{\mu\nu})$ be a normed direct system of unital algebras and $\phi_{\mu\nu}$ unit preserving (i.e. ***unital normed direct system****). Let A be the normed direct limit and $a \in A_\mu$. If $\Phi_\mu(a) \in A$ is invertible, then for all $\lambda \geq \mu$, there is a $\nu \geq \lambda'$, such that $\phi_{\mu\nu}(a) \in A_\nu$ is invertible in A_ν.*

Remark 3.2.9.
The idea behind the peculiar formulation "for all $\lambda \geq \mu$, there is a $\nu \geq \lambda'$" is, that there is not only one large ν, such that $\phi_{\mu\nu}(a) \in A_\nu$ is invertible in A_ν, but that one can find arbitrarily large ν's that yield the invertibility.

Proof 3.2.10.
Being invertible means, that there is a $c \in A$, such that

$$c\Phi_\mu(a) = \mathbb{1} = \Phi_\mu(a)c \ .$$

By corollary 3.1.12, there are $\lambda \in \mathbb{N}$ and $b \in A_\lambda$, such that $c = \Phi_\lambda(b)$. Let Œ $\lambda \geq \mu$, otherwise choose $b' = \phi_{\lambda\lambda'}(b)$, which is possible because of

$$c = \Phi_\lambda(b) = \Phi_{\lambda'}(\phi_{\lambda\lambda'}(b)) = \Phi_{\lambda'}(b') \ .$$

With $\Phi_\mu(a) = \Phi_\lambda(\phi_{\mu\lambda}(a))$, it follows that

$$\|\mathbb{1} - \Phi_\lambda(\phi_{\mu\lambda}(a))\Phi_\lambda(b)\|' = \|\mathbb{1} - \Phi_\lambda(b)\Phi_\lambda(\phi_{\mu\lambda}(a))\|' = 0 \ .$$

By definition of $\|\cdot\|'$ we need to consider $\|\phi_{\lambda\rho}(\mathbb{1} - \phi_{\mu\lambda}(a)b)\|$ for large ρ. Since $\phi_{\mu\lambda}(\mathbb{1}) = \mathbb{1}$ and $\phi_{\lambda\nu} \circ \phi_{\mu\lambda} = \phi_{\mu\nu}$:

$$\|\phi_{\lambda\rho}(\mathbb{1} - \phi_{\mu\lambda}(a)b)\| = \|\phi_{\lambda\rho}(\mathbb{1}) - \phi_{\lambda\rho}(\phi_{\mu\lambda}(a))\phi_{\lambda\rho}(b)\|$$

$$= \|\mathbb{1} - \phi_{\mu\rho}(a)\phi_{\lambda\rho}(b)\|$$

And similarly

$$\|\phi_{\lambda\rho}(\mathbb{1} - b\phi_{\mu\lambda}(a))\| = \|\mathbb{1} - \phi_{\lambda\rho}(b)\phi_{\mu\rho}(a)\| \ .$$

Both terms tend to zero for $\rho \to \infty$, such that for every $\varepsilon > 0$ there is a $\nu \geq \lambda$, such that

$$\max\left(\|\mathbb{1} - \phi_{\mu\nu}(a)\phi_{\lambda\nu}(b)\|, \|\mathbb{1} - \phi_{\lambda\nu}(b)\phi_{\mu\nu}(a)\|\right) \leq \varepsilon \ .$$

Choosing $\varepsilon \leq 1$, theorem 1.4.20 shows that $\phi_{\mu\nu}(a)\phi_{\lambda\nu}(b)$ and $\phi_{\lambda\nu}(b)\phi_{\mu\nu}(a)$ are invertible in A_ν. Thus, also $\phi_{\mu\nu}(a)$ is invertible in A_ν. □

Lemma 3.2.11.
Let $(\{A_\mu\}_{\mu\in\mathbb{N}}, \phi_{\mu\nu})$ be a normed direct system and A the normed direct limit. If all A_μ are local Banach algebras then A is a local Banach algebra.

Proof 3.2.12.
Normed direct limits and unitalization commute. Also, promotion to a matrix algebra of finite size and normed direct limit commute, by defining the maps component wise. Hence, we only need to show that A is closed under holomorphic functional calculus. And we can assume $(\{A_\mu\}_{\mu\in\mathbb{N}}, \phi_{\mu\nu})$ to be a unital normed direct system.

Let $a \in A$ and f holomorphic on an open neighborhood U of $\sigma_{\widehat{A}}(a)$. Let $b \in A_\mu$, such that $\Phi_\mu(b) = a$ and let D be the closed disk, that contains U and has radius

$$R > \limsup_\nu \|\phi_{\mu\nu}(b)\| = \|\Phi_\mu(b)\|' \geq \|a + N\|_\sim \equiv \|a\| \ .$$

Choose $z \in D \setminus U$, then $z\mathbb{1} - a$ is invertible in A by definition of U. Then by lemma 3.2.8, there is a $\nu \geq \mu$, such that $z\mathbb{1} - \phi_{\mu\nu}(b)$ is invertible in A_ν. Consider the map

$$g\colon \mathbb{C} \longrightarrow A_\nu \ , \quad w \longmapsto w\mathbb{1} - \phi_{\mu\nu}(b) \ .$$

With corollary 1.4.22 and lemma 3.2.4 it follows that $A_\nu^\times = A_\nu \cap \widehat{A}_\nu^\times$ is open. Since g is continuous, $g^{-1}(A_\nu^\times)$ is open. Thus, there is an open neighborhood V of $z \in g^{-1}(A_\nu^\times)$, such that $w\mathbb{1} - \phi_{\mu\nu}(b)$ is invertible in A_ν for $w \in V$.

Since $D \setminus U$ is compact, every open cover has a finite subcover, i.e. there are only finitely many V. Hence, there is a $\nu \geq \mu$, such that $w\mathbb{1} - \phi_{\mu\nu}(b)$ is invertible in A_ν for all $w \in D \setminus U$. Put differently, $u - \phi_{\mu\nu}(b)$ is not invertible for $u \in U$:

$$\sigma_{A_\nu}(\phi_{\mu\nu}(b)) \subset U \ .$$

By assumption, A_ν is a local Banach algebra and thus $f(\phi_{\mu\nu}(b)) \in A_\nu$. Also, by assumption, f is holomorphic on U and thus is analytical on U. Writing f as power series, and since Φ_ν is a normed morphism, i.e. continuous, one sees that f and Φ_ν commutes:

$$f(a) = f(\Phi_\mu(b)) = f(\Phi_\nu(\phi_{\mu\nu}(b))) = \Phi_\nu(f(\phi_{\mu\nu}(b))) \in A \ .$$

Hence, A is closed under holomorphic functional calculus. □

Theorem 3.2.13.
Let $(\{A_\mu\}_{\mu\in\Lambda}, \phi_{\mu\nu})$ be a normed direct system and A the normed direct limit. If all A_μ are local C^*-algebras then A is a local C^*-algebra.

Proof 3.2.14.
From lemma 3.2.11 it follows that A is already a local Banach algebra. It remains to show, that the norm $\|\cdot\|'$ satisfies the C^*-property.

Let $a \in A$, then $a = \Phi_\mu(b)$. Since all A_ν are local C^*-algebras, it holds that

$$\|\phi_{\mu_\nu}(b)^*\phi_{\mu\nu}(b)\| = \|\phi_{\mu\nu}(b)\|^2 \ .$$

Hence:

$$\|a^*a\|' = \limsup_\nu \|\phi_{\mu_\nu}(b)^*\phi_{\mu\nu}(b)\| = \limsup_\nu \|\phi_{\mu\nu}(b)\|^2 = \|a\|^2 \ .$$

□

Definition 3.2.15.
Let A be a local C^*-algebra. Consider the direct system over $\mathbb{N}$, that is defined by the sequence

$$\phi_{n,n+1} \colon M_n(A) \longrightarrow M_{n+1}(A) \ , \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \ .$$

The algebraic direct limit $M_\infty(A)$ is called **infinite matrix algebra**.

The $\phi_{n,n+1}$ define a direct system by defining the ϕ_{mn} inductively:

$$\phi_{mn} = \phi_{m,m+1} \circ \phi_{m+1,m+2} \circ \ldots \circ \phi_{n-1,n} \ .$$

Corollary 3.2.16.
The limit semi norm $\|\cdot\|'$ is a C^*-norm on $M_\infty(A)$. Hence $M_\infty(A)$ is a local C^*-algebra.

Proof 3.2.17.
The maps ϕ_{mn} of the direct system are injective $*$-morphisms and by theorem 2.6.10 isometries. Thus, for $a = \Phi_m(b) \in M_\infty(A)$, with $b \in M_m(A)$, it holds that

$$\|a\|' = \limsup_{n\in\mathbb{N}} \|\phi_{mn}(b)\| = \limsup_{n\in\mathbb{N}} \|b\| = \|b\| \ .$$

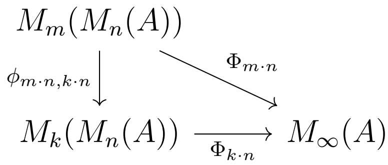
Since $\|\cdot\|$ is already a proper norm, so is $\|\cdot\|'$ here. Hence $M_\infty(A)$ is also the normed direct limit. By theorem 3.2.13, $M_\infty(A)$ is a local C^*-algebra. □

Lemma 3.2.18.
It holds that

$$\lim_{\to m} M_m(A) \cong \lim_{\to m} M_m(M_n(A)) \ .$$

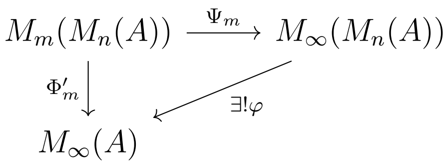
Proof 3.2.19.
First, we observe that $M_m(M_n(A)) = M_{m \cdot n}(A)$. We are considering two different direct systems $(M_m(A), \phi_{mk})$ and $(M_n(M_m(A)), \phi'_{mk})$, where $\phi'_{mk} = \phi_{m \cdot n, k \cdot n}$.

Let $(M_\infty(A), \Phi_m)$ be the direct limit of $(M_m(A), \phi_{km})$, then the following diagram commutes for $k \geq m$:



So with $\Phi'_m := \Phi_{m \cdot n}$, the tuple $(M_\infty(A), \Phi'_m)$ satisfies the mapping property of direct limits for $(M_n(M_m(A)), \phi'_{mk})$.

Let now $(M_\infty(M_n(A)), \Psi_m)$ be the direct limit of $(M_m(M_n(A)), \phi'_{km})$, then, by definition, there exists a unique morphism $\varphi \colon M_\infty(M_n(A)) \to M_\infty(A)$, such that the following diagram commutes:



It remains to show, that φ is bijective:
Injectivity: The ϕ_{mn} are injective and thus, by construction, the ϕ'_{mn} are injective as well. From corollary 3.1.15 it follows that Φ_m and so Φ'_m as well as Ψ_m are injective.

Assume now that $\varphi(a) = 0$ for $a \in M_\infty(M_n(A))$. Œ there is a $a_m \in M_m(M_n(A))$, such that $a = \Psi_m(a_m)$. From the injectivity of Φ'_m it follows that

$$0 = \varphi(\Psi_m(a_m)) = \Phi'_m(a_m) \quad \Rightarrow \quad a_m = 0 \quad \Rightarrow \quad a = \Psi_m(a_m) = 0 \ ,$$

which shows injectivity.
Surjectivity: Let $b \in M_\infty(A)$, then Œ there is an $b'_m := b_{m \cdot n} \in M_{m \cdot n}(A) = M_m(M_n(A)) =$, such that

$$b = \Phi_{m \cdot n}(b_{m \cdot n}) = \Phi'_m(a_{m \cdot n}) \equiv \Phi'_m(b'_m) \ .$$

Hence

$$b = \Phi'_m(b'_m) = \varphi(\Psi_m(b'_,)) \ ,$$

which shows surjectivity. □

Definition 3.2.20.
The completion of $M_\infty(A)$ is denoted by $A \otimes \mathcal{K}$ and called **stabilization** of A. A C^*-algebra is called **stable**, if it is isomorphic to its stabilization.

The notation for stabilization has the following reason:

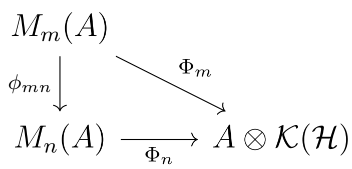
Lemma 3.2.21.
Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{K}(\mathcal{H})$ denote the compact operators on $\mathcal{H}$. Then it holds that

$$A \otimes \mathcal{K} \cong A \otimes \mathcal{K}(\mathcal{H}) \ .$$

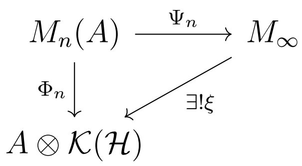
Proof 3.2.22.
Let $|n\rangle$ be a Hilbert basis of $\mathcal{H}$. Consider the injective map

$$\Phi_n \colon M_n(A) \hookrightarrow A \otimes \mathcal{K}(\mathcal{H}) \ , \quad (a_{ij}) \longmapsto \sum_{1 \leq i,j \leq n} a_{ij} \otimes |i-1\rangle\langle j-1| \ .$$

One readily checks, that Φ_n is a $*$-morphism. Consider the direct system $(M_n(A), \phi_{mn})$ of matrix algebras from definition 3.2.15, then



commutes. Furthermore, since $M_\infty(A)$ is the direct limit of $(M_n(A), \phi_{mn})$, there is a unique $*$-morphism $\xi \colon M_\infty \to A \otimes \mathcal{K}(\mathcal{H})$, such that



commutes. The Φ_n and Ψ_n are injective (see corollary 3.1.15). For every $a \in M_\infty(A)$, there is a $a_n \in M_n(A)$, such that $\Psi_n(a_n) = a$. Let $a, b \in M_\infty(A)$ with Œ $a_n, b_n \in M_n(A)$, such that $\Psi_n(a_n) = a$ and $\Psi_n(b_n) = b$ (use ϕ_{nk} otherwise). Assume that $\xi(a) = \xi(b)$, then, from the injectivity of Φ_n it follows:

$$\xi(a) = \xi(b) \quad \Leftrightarrow \quad \xi(\Psi_n(a_n)) = \xi(\Psi_n(b_n))$$
$$\Leftrightarrow \quad \Phi_n(a_n) = \Phi_n(b_n) \quad \Leftrightarrow \quad a_n = b_n$$
$$\Leftrightarrow \quad a = \Psi_n(a_n) = \Psi_n(b_n) = b \ .$$

Hence ξ is injective. Because of $\xi \circ \Psi_n = \Phi_n$, the surjectivity of Ψ_n and the fact, that Φ_n is a map on the sum of Hilbert basis elements, it follows ξ is a map on an arbitrary sum of Hilbert basis elements. Thus $\mathrm{Im}(\xi)$ is dense. Since ξ is injective, it is an isometry (see theorem 2.6.10). By the bounded linear transformation theorem, the extension $\widehat{\xi}\colon \widehat{M_\infty(A)} \to A \otimes \mathcal{K}(\mathcal{H})$ remains an isometry. So $\widehat{\xi}$ is still injective. Furthermore, since $\mathrm{Im}(\xi)$ is dense, so is $\mathrm{Im}(\widehat{\xi})$. $\widehat{M_\infty(A)}$ as C^*-algebra is closed, and thus $\mathrm{Im}(\widehat{\xi}) \in A \otimes \mathcal{K}(\mathcal{H})$ is closed by corollary 2.6.12. Yet $\mathrm{Im}(\widehat{\xi})$ was dense, so $\mathrm{Im}(\widehat{\xi}) \in A \otimes \mathcal{K}(\mathcal{H})$. Summarizing, $\widehat{\xi}$ is a bijective $*$-morphism, i.e. an isomorphism

$$\widehat{\xi}\colon \widehat{M_\infty(A)} \equiv A \otimes \mathcal{K} \xrightarrow{\cong} A \otimes \mathcal{K}(\mathcal{H}) \ .$$

□

Let G be a topological group, with norm topology. Then, we write G_0 for the connected component of the unit element of the group.

Remark 3.2.23.
Let $H \subset G$ be subgroup of $(G, \circ)$ that contains the unit element $\mathbb{1} \in H$. Let $B_\varepsilon(\mathbb{1}) \subset H$ and $h \in H$. Then $h \circ B_\varepsilon(\mathbb{1}) \subset H$ and $h \circ B_\varepsilon(\mathbb{1}) = B_\varepsilon(h)$. Hence H is open. For the same reason, the cosets $g \circ H$ are open in G.
Let $x \in G \setminus H$, then xH is a neighborhood of x. Assume that $xh \in H$, then $(xh)h^{-1} = x \in H$, which is a contradiction. Thus $xH \cap G \setminus H = \emptyset$, which means that H is also closed.

Lemma 3.2.24.
Let A be a unital local Banach algebra and $A^\times$ the set of invertible elements. Then $A_0^\times$ is the subgroup of $A^\times$, generated by the elements e^x for $x \in A$. If A is a local C^*-algebra, then $U(A)_0$ is generated by the elements e^x, where $x \in A$ such that $x^* = -x$.

Proof 3.2.25.
For $x \in A$ it holds that $e^x \in A^\times$. Furthermore, $t \mapsto e^{tx}$ describes a path from $\mathbb{1}$ to e^x, hence for all $x \in A$ it holds $e^x \in A^\times$. The Taylor series of the ln-function has a radius of convergence of 1 around 1. Thus, the open Ball with radius 1 around $\mathbb{1}$, $B_1(\mathbb{1})$ is in the image of the exp-function.

This means, that $B_1(\mathbb{1}) \subset H$, where H is the subgroup generated by $\exp(A)$. Hence, by remark 3.2.23, H is open and closed at the same time, i.e. clopen. A result from topology is, that the only clopen sets of a connected set are $\emptyset$ and the set itself. Hence $\exp(A) = A_0^\times$.

The claim for local C^*-algebras is proven in the same way. □

Corollary 3.2.26.
Let $\phi\colon A \to B$ be a unital, surjective, bounded morphism of unital local Banach algebras. Then it holds that $\phi(A_0^\times) = B_0^\times$.
If A and B are local C^*-algebras and ϕ a $*$-morphism, then $\phi(U(A)_0) = U(B)_0$.

Proof 3.2.27.
The exponential function has a Taylor series with infinite radius of convergence. Hence, continuous morphisms ϕ and exp commute. Since $A_0^\times = \exp(A)$ and $B_0^\times = \exp(B)$, the claim follows. Similarly for local C^*-algebras. □

3.3 Equivalence of idempotents and projections

Motivated by the characterization of projections (remark 2.9.19) we define:

Definition 3.3.1.
Let A be a $*$-algebra. A **projection** is an element $p \in A$ with $p^2 = p = p^*$. The set of projections is denoted by $\mathrm{Proj}(A)$.

If the algebra is not a $*$-algebra, i.e. lacking a notion of p^*, one can still use the condition of $p^2 = p$:

Definition 3.3.2.
Let A be an algebra. An **idempotent** is an element $e \in A$, such that $e^2 = e$. The set of idempotents is denoted by $\mathrm{Idem}(A)$.

Note, that the projections in a $*$-algebra are the self adjoint idempotents.

Lemma 3.3.3.
Let $e \in \mathrm{Idem}(A)$, then it holds that $\sigma_{\widetilde{A}}(e) = \{0, 1\}$.

Proof 3.3.4.
We calculate:

$$e(e - \lambda\mathbb{1}) = e - \lambda e = (1 - \lambda)e \ ,$$

$$\text{and} \quad (\mathbb{1} - e)(e - \lambda\mathbb{1}) = (\mathbb{1} - e)e - (\mathbb{1} - e)\lambda = -\lambda(\mathbb{1} - e) \ .$$

Define $a := \frac{1}{1-\lambda}e - \frac{1}{\lambda}(\mathbb{1} - a)$, which is well defined for $\lambda \not\in \{0, 1\}$. Then:

$$a(e - \lambda\mathbb{1}) = \ldots = \mathbb{1} = (e - \lambda\mathbb{1})a \ .$$

But this means that $e - \lambda\mathbb{1}$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0, 1\}$. □

Definition 3.3.5.
Let e and f be idempotents. We say that they are:

algebraically equivalent $(e \sim g)$**,** if there are $x, y \in A$, such that $e = xy$ and $f = yx$

similar ($e \sim_s f$), if there is an invertible element $z \in \widetilde{A}$ in the unitalization, such that $zez^{-1} = f$.

homotopic ($e \sim_h f$), if A is normed and there is a norm continuous path

$$e_t \colon [0,1] \longrightarrow \mathrm{Idem}(A),$$

such that $e_0 = e$ and $e_1 = f$. The path e_t is called **homotopy**.

The name "algebraically equivalent" can be understood frostudy musicm the following implication:

$$e \sim f \qquad \Rightarrow \qquad ye = yxy = fy \quad \text{and} \quad ex = xyx = xf \ .$$

Definition 3.3.6.
Let p and q be projections of a C^*-algebra. We say they are:

Murray-von Neumann equivalent ($p \sim_M q$), if there is a partial isometry $u \in A$, such that $p = u^*u$ and $q = uu^*$.

unitary equivalent ($p \sim_u q$), if there is a $u \in U(\widetilde{A})$, such that, $upu^* = q$, where $U(\widetilde{A})$ denotes the set of unitary elements in $\widetilde{A}$.

As with algebraically equivalent idempotents, MvN equivalence implies:

$$p \sim_M q \qquad \Rightarrow \qquad pu^* = u^*uu^* = u^*q \quad \text{and} \quad up = uu^*u = qu \ .$$

Although some of the following statements will be true for local Banach algebras, we assume A to be a local C^*-algebra for simplicity, if not stated otherwise.

Remark 3.3.7.
The relations $\sim_s$, $\sim_h$ and $\sim_u$ are equivalence relations.Furthermore, the following implications hold:

$$\sim_u \Longrightarrow \sim_s \qquad \text{and} \qquad \sim_M \Longrightarrow \sim \ .$$

3.3.1 Results for idempotents

Lemma 3.3.8.
Let $e, f \in \mathrm{Idem}(A)$, with $e \sim f$. Then there are $x, y \in A$, such that all of the following identities are true at the same time:

$$e = xy, \quad f = yx \ , \quad x = exf \quad \textit{and} \quad y = fye \ ,$$

Then it also holds that:

$$x = ex = xf \quad \textit{and} \quad y = fy = ye \ .$$

Proof 3.3.9.
$e \sim f$ means, that there are $a, b \in A$, such that $e = ab$ and $f = ba$. Choose $x := eaf$ and $y = fbe$. Then

$$exf = e^2af^2 = eaf = x \ , \qquad fye = f^2be^2 = fbe = y \ .$$

Also it holds that

$$xy = eaf^2be = eafbe = eababe = e^4 = e \ ,$$

and similarly $yx = f$. For the last equations, calculates

$$ex = e^2xf = exf = x \quad \text{and} \quad xf = exf^2 = exf = x$$

and similarly for $y = fy = ye$. □

Corollary 3.3.10.
The relation $\sim$ is an equivalence relation.

Proof 3.3.11.
Reflexivity and symmetry follow immediately. It remains to show transitivity. Let $e \sim f$ and $f \sim g$. Then, because of lemma 3.3.8 there are $x, x, z, w \in A$, such that

$$e = xy \ , \quad f = yx = zw \ , \quad g = wz \ , \quad y = fye \ ,$$

$$z = fzg \quad \text{and} \quad w = gwf \ .$$

Then:

$$(xz)(wy) = xfy = xy = e \quad \text{and} \quad (wy)(xz) = wfz = wz = g$$

which means that $e \sim g$. □

Definition 3.3.12.
Let $e, f \in \text{Idem}(A)$, then e and f are called **orthogonal** ($e \perp f$), if $ef = fe = 0$.

A special case is $e \perp (\mathbb{1} - e)$ in $\widetilde{A}$.

Corollary 3.3.13.
Let $e_i, f_i \in \text{Idem}(A)$ for $i = 1, 2$, such that $e_i \sim f_i$, $e_1 \perp e_2$ and $f_1 \perp f_2$. Then it holds that $e_1 + e_2 \sim f_1 + f_2$.

Proof 3.3.14.
Let $x_i, y_i \in A$, such that

$$e_i = x_iy_i \ , \quad f_i = y_ix_i \ , \quad x_i = e_ix_if_i \quad \text{and} \quad y_i = f_iy_ie_i \ .$$

Then for $i \neq j$ it holds that

$$x_i y_j = x_i f_i f_j y_j = 0$$

and similarly $y_i x_j = 0$. Hence

$$\begin{aligned} & (x_1 + x_2)(y_1 + y_2) = x_1 y_1 + x_2 y_2 = e_1 + e_2 \\ \text{and} \quad & (y_1 + y_2)(x_1 + x_2) = y_1 x_1 + y_2 x_2 = f_1 + f_2 \end{aligned}$$

$$\Rightarrow \quad e_1 + e_2 \sim f_1 + f_2 \ .$$

□

Theorem 3.3.15.
It holds that $e \sim_s f$, if and only if $e \sim f$ as well as $\mathbb{1} - e \sim \mathbb{1} - f$.

Proof 3.3.16.
⇒: It holds that $f = zez^{-1}$ for $z \in \widetilde{A}$. Let $x := ez^{-1}$ and $y := ze$, then it follows that

$$xy = ez^{-1}ze = e^2 = e \quad \text{and} \quad yx = ze^2z^{-1} = zez^{-1} = e \ .$$

Hence $e \sim f$. From $z(\mathbb{1} - e)z^{-1} = \mathbb{1} - f$ it follows that also $\mathbb{1} - e = \mathbb{1} - f$.
⇐: Let $x, y \in A$ and $a, b \in \widetilde{A}$, such that

$$e = xy \ , \quad f = yx \ , \quad x = exf \ , \quad y = fye \ ,$$

$$\mathbb{1} - e = ab \ , \quad \mathbb{1} - f = ba \ , \quad a = (\mathbb{1} - e)a(\mathbb{1} - f) \ , \quad b = (\mathbb{1} - f)y(\mathbb{1} - e) \ .$$

It holds that

$$xb = xf(\mathbb{1} - f)b = 0 \quad \text{and} \quad ya = ye(\mathbb{1} - e)a = 0 \ ,$$

$$\Rightarrow \quad (x + a)(y + b) = xy + ab = e + \mathbb{1} - e = \mathbb{1} \ .$$

Similarly one finds $(y + b)(x + a) = \mathbb{1}$. Hence $z = (x + a) \in \widetilde{A}$ is invertible with inverse $z^{-1} = (y + b)$. It holds that

$$af = (\mathbb{1} - e)a(\mathbb{1} - f)f = 0 \quad \text{and} \quad fb = 0 \ .$$

Thus we find

$$zfz^{-1} = (x + a)f(y + b) = xfy = xy = e \ .$$

Hence, $e \sim_s f$. □

Theorem 3.3.17.

Let $e \sim f$, then it holds that $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_s \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ *in* $M_2(A)$.

Proof 3.3.18.
Let $x, y \in A$, such that

$$e = xy \ , \quad f = yx \ , \quad x = exf \ , \quad y = fye \ ,$$

and define

$$z := \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix} \begin{pmatrix} \mathbb{1} - e & e \\ e & \mathbb{1} - e \end{pmatrix} \ ,$$
$$w := \begin{pmatrix} \mathbb{1} - e & e \\ e & \mathbb{1} - e \end{pmatrix} \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix} \ .$$

With $x = ex = xf$ and $y = ye = fy$, it holds that

$$x(\mathbb{1} - f) = (\mathbb{1} - e)x = 0 = y(\mathbb{1} - e) = (\mathbb{1} - f)y \ ,$$

$$\Rightarrow \quad zw = \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix}^2 = \begin{pmatrix} \mathbb{1} - f + yx & 0 \\ 0 & xy + \mathbb{1} - e \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \ .$$

In the same way, one sees that $wz = \operatorname{diag}(\mathbb{1}, \mathbb{1})$, which means that $w = z^{-1}$. A direct calculation shows that:

$$z \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} z^{-1} = \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix}$$
$$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} - f & y \\ x & \mathbb{1} - e \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \ .$$

□

Lemma 3.3.19.
Let $e, f \in \operatorname{Idem}(A)$, such that $\|e - f\| < \frac{1}{\|2e - \mathbb{1}\|}$. Then it holds that $e \sim_s f$ and $e \sim_h f$.

Proof 3.3.20.
Define $v := \mathbb{1} + (2e - \mathbb{1})(2f - \mathbb{1})$ and $z := \frac{1}{2}v$. Then

$$1 - (2e - \mathbb{1})(e - f) = (2ef - e - f + 1) = \frac{1}{2}v = z \ .$$

By assumption it holds that

$$\|(2e - \mathbb{1})(e - f)\| \leq \|2e - 1\| \cdot \|e - f\| < \|2e - 1\| \frac{1}{\|2e - 1\|} = 1 \ ,$$

such that z is invertible in $\widetilde{A}$ (theorem 1.4.20). We calculate:

$$ez = e(2ef - e - f + 1) = 2ef = (2ef - e - f + 1)f = zf$$

$$\Rightarrow \quad zfz^{-1} = e \quad \Rightarrow \quad e \sim_s f \ .$$

Define a path from $\mathbb{1}$ to z by

$$w_t := tz + \mathbb{1} - t\mathbb{1} \in \widetilde{A}^\times$$

and let $e_t = w_t^{-1} e w_t$. Then $e_t \in \mathrm{Idem}(A)$ and

$$e_0 = \mathbb{1}^{-1} e \mathbb{1} = e \quad \text{and} \quad e_1 = z^{-1} e z = f \ .$$

Thus $e \sim_h f$. □

3.3.2 Results for projections

Lemma 3.3.21.
Let $p, q \in \mathrm{Proj}(A)$, then the following statements are equivalent:

i) $p \leq q$.

ii) $\exists\ \lambda \in \mathbb{R}_{>0}\colon\ p \leq \lambda q$.

iii) $pq = qp = p$.

iv) $q - p \in \mathrm{Proj}(A)$.

Proof 3.3.22.

i) ⇔ ii):

The direction "⇒" is immediate. For the opposite direction, the case $\lambda \leq 1$ is also immediate. So let $\lambda > 1$ and consider p and q in $\widehat{A}$, which is a proper C^*-algebra.

By theorem 2.4.20, the function $f(r) = \sqrt{r}$ is operator monotone increasing. So from $p \leq \lambda q$ it follows that

$$\sqrt{p} \leq \sqrt{\lambda}\sqrt{q} \ .$$

With $p = p^2$ it follows that $\sqrt{p} = \sqrt{p^2} = p$ and in the same way $q = \sqrt{q}$, it follows that

$$p \leq \sqrt{\lambda} q \ .$$

Repeating this step leads to

$$p \leq \lambda^{\frac{1}{2^n}} q \quad \forall\ n \in \mathbb{N} \ .$$

Since A_+ is closed (theorem 2.4.9), the limit of $\lambda^{\frac{1}{2^n}} q - p \geq 0$ for $n \to \infty$ is also in $\widehat{A}_+$. This means:

$$q - p \geq 0 \quad \Leftrightarrow \quad p \leq q \ .$$

Since $p, q \in A$, this does also hold in A.

i) ⇒ iii):
Since q is a projection, it holds that $\|q\| \leq 1$ and because of lemma 2.4.11 $q \leq \mathbb{1}$:

$$p \leq q \leq \mathbb{1} \ .$$

Conjugation with p leads to

$$p = p \leq pqp \leq p \quad \Rightarrow \quad p = pqp \ .$$

Next we calculate:

$$\begin{aligned} \|qp - p\|^2 &= \|(qp-p)^*(qp-p)\| = \|(pq-p)(qp-p)\| \\ &= \|pqp - pqp - pqp + p\| = \|pqp - p\| \\ &= \|p - p\| = 0 \ . \end{aligned}$$

Thus $p = qp$ and so $p = p^* = p^*q^* = pq$.

iii) ⇒ iv):
The $*$-map is linear,such that

$$(q-p)^* = q^* - p^* = q - p \ .$$

With property iii) it follows that

$$(q-p)^2 = q^2 + p^2 - pq - qp = q + p - p - p = q - p \ .$$

Hence $q - p$ is a projection.

iv) ⇒ i):
Let r be a projection, then $r^2 = r$. Consider the function $f(t) = t^2$. Then

$$\sigma_{\widetilde{A}}(r) = \sigma(r^2) = \sigma(f(r)) = f(\sigma(r)) \subset \mathbb{R}_{\geq 0}.$$

This shows that $r \geq 0$. Hence, since $q - p$ is a projection:

$$q - p \geq 0 \quad \Leftrightarrow \quad p \leq q \ .$$

□

Theorem 3.3.23.
Let $p, q \in \mathrm{Proj}(A)$. Then $p \sim q$ if and only if $p \sim_M q$. In particular this means that $\sim_M$ is an equivalence relation.

Proof 3.3.24.
Let $x, y \in A$, such that $p = xy$, $q = yx$, $x = pxq$ and $y = qyp$. Using lemma 2.4.11 and $\|x^*x\| = \|x\|^2$ it holds that

$$p = p^*p = y^*x^*xy \leq \|x\|^2 y^*y \ .$$

In pAp the unit element is $\mathbb{1}_{pAp} = p$, so the inequality reads by definition of positive elements:

$$0 \leq \|x\|^2 y^* y - \mathbb{1}_{pAp} \quad \Leftrightarrow \quad \sigma_{pAp}(\|x\|^2 y^* y - \mathbb{1}_{pAP}) \subset [0, \infty)$$
$$\Leftrightarrow \quad \sigma_{pAp}(\|x\|^2 y^* y) \subset \left[\tfrac{1}{\|x\|^2}, \infty\right) \ .$$

By definition of the spectrum, this means that $y^*y - z\mathbb{1}_{pAp}$ is invertible for $z = 0$, i.e. y^*y is invertible in pAp.This is well defined, since

$$y^*y = (qyp)^*(qyp) = pyqyp \in pAp \ .$$

The function $\frac{1}{|r|}$ has a multiplicative inverse for $r \neq 0$ and is continuous on $\sigma_{pAp}(y)$. With lemma 2.3.12 this means that there is an $r \in pAp$, such that $r|y| = |y|r = \mathbb{1}_{pAp} = p$. Since $r \in pAp$ it also holds that $r = prp$. Furthermore:

$$p = p^* = r^*|y|^* = r^*|y| = |y|r^* \quad \Rightarrow \quad r = r^* \ .$$

Define $u := yr$, then it holds that

$$u^*u = ry^*yr = r|y|^2 r = p^2 = p \ ,$$

i.e. u^*u and uu^* are projections, because of lemma 2.9.31. It also holds that

$$u|y| = yr|y| = yp \ ,$$

where we used lemma 3.3.8 in the last step.

On the one hand:

$$quu^* = qyr^2y^* = yr^2y^* = uu^* \quad \text{and} \quad uu^*q = yr^2y^*q = yr^2y^* = uu^* \ .$$

By lemma 3.3.21, this means that $uu^* \leq q$. On the other hand

$$q = qq^* = yxx^*y^* \leq \|x\|^2 yy^* = \|x\|^2 u|y|^2u^* = \|x\|^2 uy^*yu$$
$$\leq \|x\|^2\|y\|^2 uu^* \ ,$$

where we used that $y^*y \leq \|y^*y\|\mathbb{1} = \|y\|^2\mathbb{1}$. Again, by lemma 3.3.21 this means $q \leq uu^*$

But then $q = uu^*$.

□

Theorem 3.3.25.
Let $p, q \in \mathrm{Proj}(A)$ with $p \sim_u q$, then it holds that $p \sim_M q$.

Proof 3.3.26.
By assumption it holds that $upu^* = q$ for a $u \in \tilde{A}$. By lemma 2.1.7, A is an ideal in $\tilde{A}$, such that $v = up \in A$. We calculate

$$v * v = pu * up = p^2 = p \quad \text{and} \quad vv^* = uppu^* = upu^* = q \ ,$$
$$\Rightarrow \qquad p \sim_M q \ .$$

□

Theorem 3.3.27.
Let $p, q \in \mathrm{Proj}(A)$, then $p \sim_s q$ if and only if $p \sim_u q$.

Proof 3.3.28.
⇒: Let $q = zpz^{-1}$ with $z \in \widetilde{A}^\times$. Then $zp = qz$ and thus $pz^* = z^*q$. It follows that $pz^*z = z^*qz = z^*zp$, and so $p\sqrt{z^*z} = \sqrt{z^*z}p$.

$$p = p \quad \Rightarrow \quad zp = zp \quad \Rightarrow \quad zp(z^*z)^{-\frac{1}{2}}\sqrt{z^*z}p = qz$$
$$\Rightarrow \quad z(z^*z)^{-\frac{1}{2}}p(z^*z)^{-\frac{1}{2}} = qz$$
$$\Rightarrow \quad z(z^*z)^{-\frac{1}{2}}p = qz(z^*z)^{-\frac{1}{2}}$$

Define $u = z(z^*z)^{-\frac{1}{2}}$. With $f(a)^* = \overline{f}(a)$ it follows that ($\sqrt{\cdot}$ is real for normal elements):

$$\begin{aligned} uu^* &= z(z^*z)^{-\frac{1}{2}}\left((z^*z)^{-\frac{1}{2}}\right)z^* = z(z^*z)^{-\frac{1}{2}}(z^*z)^{-\frac{1}{2}}z^* \\ &= z(z^*z)^{-1}z^* = zz^{-1}(z^*)^{-1}z^* = \mathbb{1} \end{aligned}$$

In the same way one calculates $u^*u = \mathbb{1}$. Hence u is unitary. So:

$$up = qu \quad \Rightarrow \quad upu^* = q \quad \Rightarrow \quad p \sim_u q\ .$$

⇒: This is trivial since $u^* = u^{-1}$.

□

Corollary 3.3.29.
Let $p, q \in \mathrm{Proj}(A)$, such that $\|p - q\| < 1$, then it holds that $p \sim_h q$.

Proof 3.3.30.
By assumption $2p - \mathbb{1}$ is self-ajoint. Furthermore, it is unitary:

$$(2p - \mathbb{1})^*(2p - \mathbb{1}) = 4p^2 - 2p - 2p + \mathbb{1} = 4p - 4p + \mathbb{1} = \mathbb{1} = (2p - \mathbb{1})(2p - \mathbb{1})^*\ .$$

Thus it holds that

$$1 = \|\mathbb{1}\| = \|(2p - \mathbb{1})^*(2p - \mathbb{1})\| = \|(2p - \mathbb{1})\|^2\ .$$

Hence $\|p - q\| < 1 = \|(2p - \mathbb{1})\| = \frac{1}{\|(2p-\mathbb{1})\|}$. The rest follows from lemma 3.3.19. □

Corollary 3.3.31.
Let $e \sim_h f$, where $e_t \in \mathrm{Idem}(A)$ denotes the homotopy. Then there is a norm continuous path $z_t \in \widetilde{A}^\times$ with $z_0 = \mathbb{1}$ and $z_t^{-1}ez_t = e_t$ for all $t \in [0, 1]$, i.e. it holds

that $e \sim_s f$.
If e and f are projections, and $e_t \in \mathrm{Proj}(A)$, then one can assume that $z_t \in U(\widetilde{A})$.

Proof 3.3.32.
Choose $M > 0$, such that $\|2e_t - \mathbb{1}\| \leq M$ for all $t \in [0,1]$. Let $0 = t_0 < t_1 < \ldots < t_n = 1$, such that $\|e_s - e_r\| < \frac{1}{M}$ for $e, r \in [t_j, t_{j+1}]$. Define:

$$\left.\begin{aligned} v_t^j &:= \mathbb{1} + (2e_{t_j} - \mathbb{1})(2e_t - \mathbb{1}) \\ u_t^j &:= \tfrac{1}{2} v_t^i \\ z_t &:= u_{t_1}^0 \ldots u_{t_j}^{j-1} u_t^j \end{aligned}\right\} \quad t \in [t_j, t_{j+1}] \ .$$

From lemma 3.3.19 it follows that $u_t^i \in \widetilde{A}^\times$, such that $z_t \in \widetilde{A}^\times$ is well defined. With

$$u_{t_j}^j = \mathbb{1} + (2e_{t_j} - \mathbb{1})(2e_{t_j} - \mathbb{1}) = \mathbb{1} + 4e_{t_j} - 2e_{t_j} - 2e_{t_j} = \mathbb{1}$$

it follows that

$$u_{t_1}^0 \ldots u_{t_{j+1}}^j = u_{t_1}^0 \ldots u_{t_{j+1}}^j u_t^{j+1}\Big|_{t=t_{j+1}} \ ,$$

which shows that z_t depends continuously on t. Similar to the prove of lemma 3.3.19, one shows that

$$(u_s^{j-1}) e_{t_{j-1}} u_s^{j-1} = e_{t_{j-1}+s} \ ,$$

$$\Rightarrow \quad z_t^{-1} e z_t = (u_t^j)^{-1} (u_{t_j}^{j-1})^{-1} \ldots (u_{t_1}^0)^{-1} e u_{t_1}^0 \ldots u_{t_{j+1}}^j u_t^{j+1} = e_t \ .$$

for $t \in [t_j, t_{j+1}]$.

□

Theorem 3.3.33.
Let $e \sim_s f$, then it holds that $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.

For the proof we follow [Bla86, proposition 3.4.1 and 4.4.1].

Proof 3.3.34.
By assumption there is a $z \in \widetilde{A}^\times$, such that $f = zez^{-1}$. Define the invertible path

$$u_t = \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

and consider $w_t = \mathrm{diag}(z,1)\ u_t\ \mathrm{diag}(z^{-1},1)\ u_t^{-1}$. One easily checks, that w_t is a homotopy for $\mathrm{diag}(\mathbb{1},\mathbb{1})$ and $\mathrm{diag}(z,z^{-1})$. Now set $e_t = w_t \mathrm{diag}(e,0) w_t^{-1}$, then:

$$e_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$e_1 = \mathrm{diag}(z, z^{-1})\ \mathrm{diag}(e,0)\ \mathrm{diag}(z^{-1}, z) = \mathrm{diag}(zez^{-1}, 0) = \mathrm{diag}(f, 0) \ .$$

□

Corollary 3.3.35.
There are paths, such that for $x, y \in A$ and $z \in \widetilde{A}^\times$ the following holds:

$$\begin{pmatrix} xy & 0 \\ 0 & \mathbb{1} \end{pmatrix} \sim_h \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \sim_h \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \sim_h \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

Furthermore, for $x, y \in U(A)$, the homotopies are in $U_2(A)$ as well, and for $z \in U(A)$ it follows that

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \sim_h \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$$

Proof 3.3.36.
Let u_t and w_t be the paths from the proof of theorem 3.3.33. Furthermore define $p_t := \mathrm{diag}(x, \mathbb{1})u_t\mathrm{diag}(y, \mathbb{1})u_t^{-1}$ and $v_t = u_t\mathrm{diag}(x, y)u_t^{-1}$, then they are the homotopies for

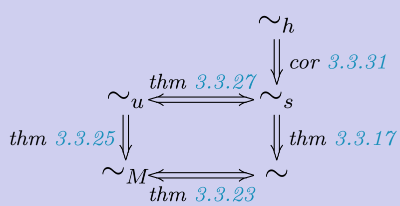
$$\begin{array}{rccc} p_t\colon & \mathrm{diag}(xy, \mathbb{1}) & \text{to} & \mathrm{diag}(x, y) \\ u_t\colon & \mathrm{diag}(x, y) & \text{to} & \mathrm{diag}(y, x) \\ w_t\colon & \mathrm{diag}(\mathbb{1}, \mathbb{1}) & \text{to} & \mathrm{diag}(z, z^{-1}) \end{array}$$

Since $u_t \in U_2(A)$ already, the paths are unitary, if x, y, z are. Furthermore, the last homotopy relation follows from $z^* = z^{-1}$. □

3.3.3 Conclusion

The results of this section are summarized in the following theorem:

Theorem 3.3.37.
Between the different relations for projections, the following implications holds:



In case of matrix algebras, the implications become equivalences (see theorem 3.3.17 and 3.3.33).

Theorem 3.3.38.
Every idempotent is homotopic to a projection in A. Furthermore, let $p, q \in \operatorname{Proj}(A)$ with $p \sim_h q$, then there is a homotopy between p and q in $\operatorname{Proj}(A)$.

Proof 3.3.39.
Let $e \in \operatorname{Idem}(A)$ and define

$$z := \mathbb{1} + (e - e^*)(e^* - e) = \mathbb{1} + (e^* - e)^*(e^* - e) \\ = \mathbb{1} - (e - e^*)^2 \ .$$

Since $(e^* - e)^*(e^* - e)$ is positive by theorem 2.4.9 and thus $z \in \widetilde{A}^\times$ is positive and self adjoint. It holds that

$$(e - e^*)^2 e = (e - e^*)(e - e^* e) = e - e^* e - ee^* e + e^* e = e - ee^* e$$

$$\text{and} \quad e(e - e^*)^2 = \ldots = e - ee^* e \ .$$

Hence $ze = ee^*e = ez$ and thus $ze^* = e^*z$. Then $p := ee^*z^{-1} = z^{-1}ee^* \in A$ and $p^* = p$. Furthermore

$$p^2 = z^{-1}ee^*ee^*z^{-1} = z^{-1}zee^*z^{-1} = ee^*z^{-1} = p \ ,$$

which shows that $p \in \operatorname{Proj}(A)$.

It holds that $ep = e^2e^*z^{-1} = ee^*z^{-1} = p$ and $pe = z^{-1}ee^*e = z^{-1}ze = e$. Let $t \in \mathbb{C}$, then

$$(\mathbb{1} + t(p - e))(\mathbb{1} - t(p - e)) = \mathbb{1} - t^2(e - p)^2 = \mathbb{1} - t^2(e - ep - pe + p) = \mathbb{1} \ .$$

Hence $u_t := \mathbb{1} + t(p - e) \in \widetilde{A}^\times$. Let $p_t := u_t^{-1} e u_t$. Then $p_0 = e$ and

$$p_1 = (\mathbb{1} - p + e)e(\mathbb{1} + p - e) = (\mathbb{1} - p + e)p = ep = p \ .$$

So p_t is a homotopy between e and p, i.e. $e \sim_h p$.

Let $p, q \in \operatorname{Proj}(A)$ and e_t a homotopy in $\operatorname{Idem}(A)$, such that $p \sim_h q$. Then, direct calculations show that $e_t^* e_t(\mathbb{1} + (e_t - e_t^*)(e_t^* - e_t))$ is a projection and a homotopy between p and q. □

Theorem 3.3.40.
Let $(A_\mu, \phi_{\mu\nu})$ be a normed direct system of local Banach algebras and A the normed direct limit. If $e \in \operatorname{Idem}(A)$, then for all $\mu \in \mathbb{N}$, there are a $\nu \geq \mu$ and an $e_0 \in \operatorname{Idem}(A_\nu)$, such that $\Phi_\nu(e_0) = e$.
If $e, f \in \operatorname{Idem}(A)$ with $e \sim_s f$ in A, then for all $\mu \in \mathbb{N}$, there are $\nu \geq \mu$ and $e_0, f_0 \in \operatorname{Idem}(A_\nu)$, such that $\Phi_\nu(e_\nu) = e$, $\Phi_\nu(f_0) = f$ and $e_0 \sim_s f_0$ in A_ν.

Proof 3.3.41.
Part 1: Let $\mu' \geq \mu$ and $a \in A_{\mu'}$, such that $e = \Phi_{\mu'}(a)$. Using lemma 3.2.8 for $e - z\mathbb{1}$ and $\Phi_\nu(a) - z\mathbb{1}$ respectively, we obtain the following statement:

$$\bigcap_{\nu \geq \mu'} \sigma_{A_\nu}(\phi_{\mu'\nu}(a)) = \sigma_A(e) = \{0, 1\} \ .$$

Thus there are $\mu'' \geq \mu'$ and open neighborhoods U, V of 0 and 1 respectively, such that $U \cap V = \emptyset$ and $\sigma_{A_\nu}(\phi_{\mu'\nu}(a)) \subset U \cup V$, for all $\nu \geq \mu''$. There is a function f with $f(U) = 0$ and $f(V) = 1$. Then $f(\phi_{\mu'\nu}(a)) \in \operatorname{Idem}(A_\nu)$, since $f(z)^2 = f(z)$ on $U \cup V$.

Part 2: Let now $e \sim_s f$ and fix a $\mu \in \mathbb{N}$. By the first part of the theorem, there is a $\mu' \geq \mu$, such that for all $\nu \geq \mu'$ there are $e_0, f_0 \in \operatorname{Idem}(A_\nu)$ with $e = \Phi_\nu(e_0)$ and $f = \Phi_\nu(f_0)$. Let $z \in \widetilde{A}^\times$, such that $zez^{-1} = f$, according to $e \sim_s f$. Let $w \in A_\nu$, such that $\Phi_\nu(w) = z$. By lemma 3.2.8, we can assume that $w \in \widetilde{A}_\nu^\times$.
Define $f_1 := we_0w^{-1}$, then:

$$\Phi_\nu(f_1) = \Phi_\nu(w)\Phi_\nu(e_0)\Phi_\nu(w)^{-1} = zez^{-1} = f = \Phi_\nu(f_0) \ .$$

By the construction of the normed direct limit, we can choose ν large enough, such that $\|f_1 - f_0\|$ becomes small enough to apply lemma 3.3.19. Then $f_1 \sim_s f_0$ and $f_1 \sim_s e_0$ by construction. Thus: $f_0 \sim_s e_0$. □

The opposite direction is also true:

Corollary 3.3.42.
Let $e, f \in \operatorname{Idem}(A_n)$ with $e \sim_s f$, then it holds that $\Phi_n(e) \sim_s \Phi_n(f)$ in A.

Proof 3.3.43.
There is a $z \in \widetilde{A}_n^\times$, such that $zez^{-1} = f$, by definition. Then, since Φ_n is a morphism, it holds that $\Phi_n(z^{-1}) = \Phi_n(z)^{-1}$. It follows that

$$\Phi_n(f) = \Phi_n(zez^{-1}) = \Phi_n(z)\Phi_n(e)\Phi_n(z)^{-1} \ ,$$

$$\Rightarrow \quad \Phi_n(e) \sim_s \Phi_n(f) \ .$$

□

We conclude this subsection with a proposition from [Bla86, Proposition 4.5.1], as its proof will be similar to the proof of theorem 3.3.40.

Lemma 3.3.44.
Let A be a local Banach algebra and $e, f \in \operatorname{Idem}(\widehat{A})$. Then for every $\varepsilon > 0$ there is an $e \in \operatorname{Idem}(A)$, such that $\|e - e_0\| < \varepsilon$.
Furthermore, if $e \sim_s f$, then there is an $f_0 \in \operatorname{Idem}(A)$, such that $\|f - f_0\| < \varepsilon$ and $e_0 \sim_s f_0$.

Proof 3.3.45.
As result of the completion, it follows that A is dense in $\widehat{A}$. Hence there is an $x \in A$, such that $\|x - e\| < \varepsilon$. Choosing ε small enough, $\sigma(x) \subset U \cup V$, where U is a neighborhood of 0 , V is a neighborhood of 1 and $U \cap V = \emptyset$. Similar to the proof of 3.3.40, we find $f(x) = e_0$.

Now consider $f \sim_s e$ in $\widehat{A}$, i.e. $f = zez^{-1}$, where z is in the unitalization of $\widehat{A}$. Let $w \in \widetilde{A}^\times$, such that $\|w - z\|$ is small enough for $f_0 := we_0w^{-1}$, i.e. $f_0 \sim_s e_0$. Since product and inversion are continuous w.r.t. the norm, it holds that

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0 \colon \forall\, e', z' \colon \begin{matrix} \|e - e'\| < \delta \\ \|z - z'\| < \delta \end{matrix} \quad \Rightarrow \quad \|z'e'z'^{-1} - zez^{-1}\| < \varepsilon\ .$$

This is used to show that $\|f - f_0\| < \varepsilon$. □

3.4 The K_0-group

In the following, A denotes a local C^*-algebra.

3.4.1 The functor $V(A)$

Definition 3.4.1.
With $V(A)$ we denote the quotient ${}^{\mathrm{Idem}(M_\infty(A))}/_{\sim}$.

From the results of subsection 3.3.3 we know, that we can equivalently choose the relations $\sim_s$ or $\sim_h$. Furthermore, we could equivalently consider $\mathrm{Proj}(M_\infty(A))$ with any of the relations, because of theorem 3.3.38.

Lemma 3.4.2 (Monoid structure).
$V(A)$ has an abelian monoid structure[1] *defined by*

$$[e] + [f] = [e' + f']$$

for $e' \in [e]$ and $f' \in [f]$, such that $e' \perp f'$, with neutral element $[0]$.

Proof 3.4.3.
Existence: Let $e \in [e]$ and $e_0 \in A_j$, such that $\Phi_j(e_0) = e$, which is possible because of theorem 3.3.40. Then:

$$e = \Phi_j(e_0) = \Phi_{j+1}(\phi_{j,j+1}(e_0)) = \Phi_{j+1}\left(\begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix}\right)\ .$$

Furthermore:

$$E_0 := \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = xy$$

[1] Loosely speaking, a monoid is a group without the invertibility requirement.

and

$$E_0' := \begin{pmatrix} 0 & 0 \\ 0 & e_0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & e_0 \\ 0 & 0 \end{pmatrix} = yx \ .$$

Hence $E_0 \sim E_0'$ and thus $\Phi_{j+1}(E_0) \sim \Phi_{j+1}(E_0')$. It follows that

$$\Phi_{j+1}(E_0') \in [e] \ .$$

Choose $f_0 \in A_j$, such that $\Phi_j(f_0) = f \in [f]$, and so

$$f_0 = \Phi_{j+1}\left(\begin{pmatrix} f_0 & 0 \\ 0 & 0 \end{pmatrix}\right) := \Phi_{j+1}(F_0) \ .$$

Then $F_0 \perp E_0'$, and since Φ_{j+1} is a morphism, $\Phi_{j+1}(F_0) \perp \Phi_{j+1}(E_0')$. Choosing $e' = \Phi_{j+1}$ and $f' = \Phi_{j+1}(F_0)$ shows the existence.

Well definedness: Let $e'' \in [e]$ and $f'' \in [f]$, such that $e'' \perp f''$. It holds that $e' \sim e''$ and $f' \sim f''$. From corollary 3.3.13 (for $e' = e_1$, $e'' = f_1$, $f' = e_2$ and $f'' = f_2$) it follows that

$$e' + f' \sim e'' \sim f'' \quad \Leftrightarrow \quad e'' + f'' \in [e' + f'] \ .$$

□

Remark 3.4.4.

i) By construction it holds that $V(A) \cong V(B)$, if $M_\infty(A) \cong M_\infty(B)$. It especially holds that $V(A) \cong V(M_n(A))$, since $M_\infty(A) \cong M_\infty(M_n(A))$, as seen in lemma 3.2.18.

ii) From lemma 3.3.44 and 3.3.19 it follows that $V(A) = {}^{\mathrm{Idem}(A\otimes\mathcal{K})}/_{\sim_s}$. Thus it holds that $V(A) \cong V(B)$ if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, i.e. A and B are **stably isomorphic**.

Corollary 3.4.5.
If A is separable, then $V(A)$ is countable

Proof 3.4.6.
Assume that $V(A)$ is uncountable. Then there is an uncountable set $P \subset A$ of projections, such that:

$$\forall p, q \in P \ , \ p \neq q \quad : \quad p \not\sim_h q \ .$$

This means that $[p] \neq [q]$. I it holds that

$$B_{\frac{1}{2}}(p) \cap B_{\frac{1}{2}}(q) = \emptyset \quad \forall \ p \neq q \ .$$

This can be seen as follows. Assume that

$$x \in B_{\frac{1}{2}}(p) \cap B_{\frac{1}{2}}(q) \quad \Rightarrow \quad \begin{array}{l} \|x-p\| < \frac{1}{2} \\ \|x-q\| < \frac{1}{2} \end{array}$$

$$\Rightarrow \quad \|p-q\| < \|x-p\| + \|x-q\| < 1 \ .$$

But from corollary 3.3.29 it would follow $p \sim_h q$, contradicting $p \neq q \quad p \not\sim_h q$. Thus $B_{\frac{1}{2}}(p) \cap B_{\frac{1}{2}}(q) = \emptyset$. So every $B_{\frac{1}{2}}(p)$ corresponds to a class $[p] \in V(A)$.

On the other hand, since A is separable, there is a dense sequence $(x_n) \subset A$. Because of the denseness,

$$\forall\ p \in E\ \exists\ n_p \in \mathbb{N} :\ x_{n_p} \in B_{\frac{1}{2}}(p) \ .$$

If the map $p \mapsto n_p$ was injective, the set P would be countable, which is not the case. So $p \mapsto n_p$ is not injective, which means, that there are $n_p = n_q$. However, then it follows $x_{n_p} \in B_{\frac{1}{2}}(p) \cap B_{\frac{1}{2}}(q) = \emptyset$, which is a contradiction. Hence the assumption $V(A)$ being uncountable is wrong. □

For the functorality of $V(a)$ we make the following observation. Let $e, e' \in [e] \in V(A)$. Because of Theorem 3.3.40, there are $e_0, e_0' \in \mathrm{Idem}(M_n(A))$, such that $\Phi_n(e_0) = e$, $\Phi(e_0') = e'$ and $e_0 \sim e_0'$. Since $\sim$ is an equivalence relation, there is an $e_0'' \in \mathrm{Idem}(M_n(A))$ for all $e'' \in [e]$, such that $\Phi_n(e_0'') = e''$ and $e_0'' \sim e_0$.

On the other hand, let $a \sim e_0$, i.e. $a \in [e_0]_n \in {}^{\mathrm{Idem}(M_n(A))}/_{\sim}$. Then there are $x, y \in M_n(A)$, such that

$$e_0 = xy \quad \text{and} \quad a = yx \ .$$

Since Φ_n is a morphism, it holds that

$$e = \Phi_n(e_0) = \Phi_n(x)\Phi_n(y) \quad \text{and} \quad \Phi_n(a) = \Phi_n(y)\Phi_n(x) \ .$$

Hence $\Phi_n(a) \sim e$ and thus $\Phi_n(a) \in [e]$. So it holds that $\Phi_n([e_0]_n) = [e] \in V(A)$, and for all

$$\forall\ [e] \in V(a)\ \forall\ e' \in [e]\ \exists\ [a] \in {}^{\mathrm{Idem}(M_n(A))}/_{\sim}\colon \quad \exists\ b \in [a] \colon \Phi_n(b) = e' \ .$$

Hence $[e]_n$, which denotes the corresponding class of $[e]$ in ${}^{\mathrm{Idem}(M_n(A))}/_{\sim}$, is well defined.

Lemma 3.4.7 (**Functoriality of V**).
Let $\phi\colon A \to B$ be a $*$-morphism. We define:

$$\phi_*\colon V(A) \longrightarrow V(B) \ , \qquad \phi_*([e]) = [\Phi_n(\phi(e_0))] \ ,$$

for an $e_0 \in [e]_n$. The map ϕ is extended component wise to $M_n(A)$:

$$\phi(e) = \phi((e_{ij})) = (\phi(e)_{ij}) \ .$$

The map ϕ_* is well defined and satisfies

$$(\mathrm{Id}_A)_* = \mathrm{Id}_{V(A)} \ , \qquad (\psi \circ \phi)_* = \psi_* \circ \phi_* \ ,$$

for any other $*$-morphism $\psi\colon B \to C$.

Proof 3.4.8.
Well definedness already follows from the construction of $[e_0]_n \in {}^{\mathrm{Idem}(M_n(A))}/_{\sim}$ and the component wise definition, since ϕ is a $*$-morphism.
For the first equation, we calculate:

$$(\mathrm{Id}_A)_*([e]) = [\Phi_n(\mathrm{Id}_A(e_0))] = [\Phi_n(e_0)] = [e] \quad \Rightarrow \quad (\mathrm{Id}_A)_* = \mathrm{Id}_{V(A)} \ .$$

For the second equation, we observe that for the class $[\Phi_n(\phi(e_0))]$, we can choose $\phi(e_0) \in [\Phi_n(\phi(e_0))]_n$. Then

$$\begin{aligned}(\psi_* \circ \phi_*)([e]) &= \psi_*([\Phi_n(\phi(e_0))]) = [\Phi_n(\psi(\phi(e_0)))] \\ &= [\Phi_n((\psi \circ \phi)(e_0))] = (\psi \circ \phi)_*([e]) \ .\end{aligned}$$

□

exm:constant direct system

Definition 3.4.9.
Let $\phi_0, \phi_1 \colon A \to B$ be $*$-morphisms of local C^*-algebras. Then ϕ_0 and ϕ_1 are called **homotopic**, if there is a $*$-morphism $\Phi \colon A \to C([0,1], B)$, such that

$$\mathrm{ev}_0 \circ \Phi = \phi_0 \quad \text{and} \quad \mathrm{ev}_1 \circ \Phi = \phi_1 \ ,$$

where $\mathrm{ev}_t \colon C([0,1], B) \to B$ is the evaluation map. The map Φ is called **Homotopy** between ϕ_0 and ϕ_1.

For the evaluation we write $\mathrm{ev}_t \circ \Phi =: \Phi_t$. This is a path of $*$-morphisms $A \to B$, since the algebra operations are defined point wise on $C([0,1], B)$, w.r.t. $t \in [0,1]$:

$$\Phi_t(a \diamond a') = \mathrm{ev}_t(\Phi(a \diamond a')) = \mathrm{ev}_t(\Phi(a) \diamond \Phi(a')) = \Phi_t(a) \diamond \Phi_t(a') \ .$$

$$\Phi_t(a^*) = \mathrm{ev}_t(\Phi(a^*)) = \mathrm{ev}_t(\Phi(a)^*) = \Phi_t(a)^*$$

Furthermore, for all $a \in A$, the map $t \mapsto \Phi_t(a)$ is a continuous map $[0,1] \to B$.
The opposite direction holds true as well:

Corollary 3.4.10.
Let $\Psi_t \colon A \to B$ be a path of $*$-morphisms for $t \in [0,1]$, such that $t \mapsto \Psi_t(a)$ is a continuous map $[0,1] \to B$ for all $a \in A$. If $\Psi_0 = \phi_0$ and $\Psi_1 = \phi_1$, then Ψ_t is a homotopy between ϕ_0 and ϕ_1.

Proof 3.4.11.
For every $a \in A$, the map $\Psi(a) \colon [0,1] \to B,\ t \mapsto \Psi_t(a)$ is continuous, such that $\Psi(a) \in C([0,1], B)$. The algebra structure is defined point wise, i.e.

$$\Psi(a) = t \mapsto \Psi_t(a \diamond a') = t \mapsto \Psi_t(a) \diamond \Psi_t(a') = \Psi(a) \diamond \Psi(a') \ ,$$

$$\Psi(a^*) = t \mapsto \Psi_t(a^*) = t \mapsto \Psi_t(a)^* = \Psi(a)^* \ .$$

Now we define $\Phi\colon A \to C([0,1], B),\ a \mapsto \Psi(a)$. The previous calculations then show that Φ is a $*$-morphism. Finally we observe that

$$(\mathrm{ev}_t \circ \Phi)(a) = \mathrm{ev}_t(\Phi(a)) = \mathrm{ev}_t(\Psi(a)) = \Psi_t(a)\ \forall\ a \in A$$

$$\Rightarrow \quad \mathrm{ev}_0 \circ \Phi = \phi_0 \quad \text{and} \quad \mathrm{ev}_1 \circ \Phi = \phi_1 \ .$$

□

Theorem 3.4.12.
The assignment $A \mapsto V(A)$, $\phi \mapsto \phi_*$ is a functor from local C^*-algebras to abelian monoids, that has the following properties:

i) V is ***homotopy invariant****, i.e. if ϕ_0 and ϕ_1 are homotopic, then $(\phi_0)_* = (\phi_1)_*$.*

ii) V is ***additive****, i.e. $V(A \oplus B) \cong V(A) \oplus V(B)$.*

iii) V commutes with direct limits over $\mathbb{N}$: Let A be the normed direct limit of a direct system of local C^*-algebras $((A_\mu)_{\mu\in\mathbb{N}}, \phi_{\mu\nu})$. Then $V(A)$ is the direct limit of $(V(A_\mu), (\phi_{\mu\nu})_*)$.

For the proof of iii) we follow [Weg93, proof of Proposition 6.2.9] closely.

Proof 3.4.13.
i) Let $e \in \mathrm{Idem}(M_n(A))$ and ϕ be a homotopy between ϕ_0 and ϕ_1. Consider $\phi_t \coloneqq \mathrm{ev}_t \circ \phi$, which is a path of $*$-morphisms $A \to B$. Then $\phi_t(e) \in \mathrm{Idem}(M_n(A))$, since

$$\phi_t(e)^2 = \phi_t(e^2) = \phi_t(e) \ .$$

Hence $\phi_0(e) \sim_h \phi_1(e)$. But then $\phi_0(e) \sim \phi_1(e)$ for any other relation too. For any $[e'] \in V[A]$ there is an $e \in \mathrm{Idem}(M_n(A))$, such that $\Phi_n(e) = e'$. But since $\phi_0(e) \sim_h \phi_1(e)$ it follows that $\Phi_n(\phi_0(e)) \sim_h \Phi_n(\phi_1(e))$, so $[\Phi_n(\phi_0(e))] = [\Phi_n(\phi_1(e))]$. Hence:

$$(\phi_0)_*([e']) = [\Phi_n(\phi_0(e))] = [\Phi_n(\phi_1(e))] = (\phi_1)_*([e']) \ .$$

ii) In the finite dimensional case $M_n(A \oplus B) \cong M_n(A) \oplus M_n(B)$ is clear. But then $(M_\infty(A \oplus B), \Phi_n)$ is the direct limit of $(M_n(A \oplus B), \phi_{mn})$ and $(M_\infty(A) \oplus M_\infty(B), \Phi'_n)$ is the direct limit of $(M_n(A) \oplus M_n(B), \phi_{mn})$. Let $\Psi_n\colon M_n(A \oplus B) \xrightarrow{\cong} M_n(A) \oplus M_n(B)$ the isomorphism. Then the following diagram commutes:

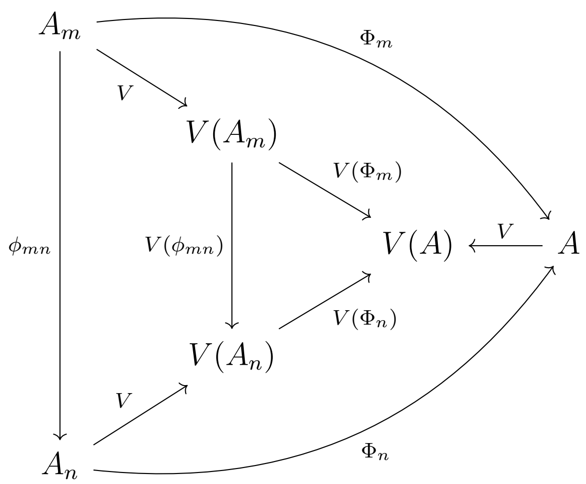
$$\begin{array}{ccc} M_m(A \oplus B) & & \\ \downarrow{\scriptstyle \phi_{mn}} & \searrow^{\Phi_m \circ \Psi_m} & \\ M_n(A \oplus B) & \xrightarrow[\Phi_n \circ \Psi_n]{} & M_\infty(A) \oplus M_\infty(B) \end{array}$$

But by the definition of direct limits and theorem 3.1.8 this means that

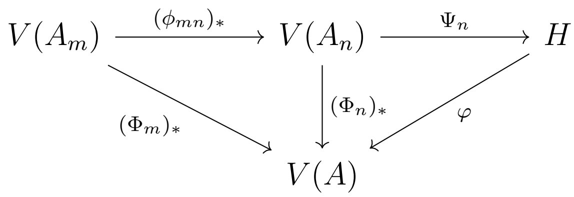
$$M_\infty(A \oplus B) \cong M_\infty(A) \oplus M_\infty(B) \ .$$

From the isomorphy in the finite dimensional case and theorem 3.3.40, it follows that the equivalence classes with respect to $\sim$ are the same for $M_\infty(A \oplus B)$ and $M_\infty(A) \oplus M_\infty(B)$. Thus the claim follows.

iii) Denote the functor by V. That $(V(A_n), \phi_{nm})$ is a direct system follows from the functoriality, by drawing a commutative diagram. As this is similar to the mapping property, we skip this step here. For the mapping property we observe, that the following diagram commutes:



The inner diagram is the mapping property of a direct limit. Let now (H, Ψ_n) be the direct limit of the direct system $(V(A_m), \phi_{mn})$. Then, there is a unique semi group morphism ξ, such that



It especially follows that $\xi \circ \Psi_n = (\Phi_n)_*$. To show that $V(A)$ is a direct limit, it is enough to show that $V(A) \cong H$. Hence, we only need to show that ξ is bijective.

For **surjectivity**, consider $p \in \mathrm{Idem}(M_\infty(A))$. By theorem 3.3.40 there is a $p_0 \in \mathrm{Idem}(M_j(A))$, such that $P_j(p_0) = p$. Here, (M_∞, P_j) is the direct system of the matrix algebras. Using theorem 3.3.40 again, there is an $n \in \mathbb{N}$, such that $\exists\ q \in M_j(A_n)\ :\ \Phi_n(q) = p_0$. Here Φ_n acts component wise. With the definition of $(\Phi_n)_*$ we see that:

$$\begin{aligned}[p] = [P_j(p_0)] = [P_j(\Phi_n(q))] = (\Phi_n)_*([P_j(q)]) = (\xi \circ \Psi_n)([P_j(q)]) \\ = \xi(\Psi_n([P_j(q)])) \ .\end{aligned}$$

This shows that $[p] \in \mathrm{Im}(\xi)$ and hence surjectivity.

To show **injectivity** consider $[P_j(p_{n_0})], [P_j(q_{n_0})] \in V(A_j)$, with $p_{n_0}, q_{n_0} \in \mathrm{Proj}(M_j(A_{n_0}))$. Assume, that both elements create the same image for ξ, i.e.

$$\begin{aligned}[P_j(\Phi_{n_0}(p_{n_0}))] &= (\Phi_{n_0})_*([P_j(p_{n_0})]) = (\xi \circ \Psi_{n_0})([P_j(p_{n_0})]) \\ &\overset{!}{=} (\xi \circ \Psi_{n_0})([P_j(q_{n_0})]) = \ldots = [P_j(\Phi_{n_0}(q_{n_0}))] \ .\end{aligned}$$

To keep the notation short, let $p := \Phi_{n_0}(p_{n_0}), q := \Phi_{n_0}(q_{n_0}) \in \mathrm{Proj}(M_j(A))$. Form $[P_j(p)] = [P_j(q)]$ it follows that $p \sim_u q$, by theorem 3.3.40.

$$\Rightarrow \quad \exists\ u \in U_j(A): \quad q = upu^* \ .$$

Choose[2] $n_1 \in \mathbb{N}$ large enough, such that there is a $u_{n_1} \in U_j(A_{n_1})$, that for $\varepsilon > 0$ it holds that

$$\|u - \Phi_{n_1}(u_{n_1})\| \leq \varepsilon \ .$$

Choose $n_2 \in \mathbb{N}$, such that $n_2 \geq n_0$ and $n_2 \geq n_1$, and define:

$$p_{n_2} := \phi_{n_0 n_2}(p_{n_0}) \ , \qquad q_{n_2} := \phi_{n_0 n_2}(q_{n_0}) \quad \text{and} \quad u_{n_2} := \phi_{n_1 n_2}(u_{n_1}) \ .$$

From the mapping property of the direct system $((A_n), \phi_{nm})$ it follows that:

$$p = \Phi_{n_0}(p_{n_0}) = \Phi_{n_2}(\phi_{n_0 n_2}(p_{n_0})) = \Phi_{n_2}(p_{n_2}) \ , \quad q = \Phi_{n_2}(q_{n_2})$$

$$\text{and} \quad \|u - \Phi_{n_2}(u_{n_2})\| = \|u - \Phi_{n_2}(\phi_{n_1 n_2}(u_{n_1}))\| = \|u - \Phi_{n_1}(u_{n_1})\| \leq \varepsilon \ .$$

Next, define the following:

$$q'_{n_2} := u_{n_2} p_{n_2} u^*_{n_2} \quad \text{and} \quad u' := \Phi_{n_2}(u_{n_2}) \ . \tag{3.1}$$

In this notation, $\|u - u'\| \leq \varepsilon$ and thus (with $\|p\| = 1$, since p is a projection):

$$\begin{aligned}\|\Phi_{n_2}(q'_{n_2}) - \Phi_{n_2}(q_{n_2}) &= \|u'pu'^* - upu^*\| \leq \|u' - u\| + \|(u' - u)^*\| \\ &= 2\|u - u'\| \leq 2\varepsilon \ .\end{aligned}$$

Hence, there is an $m \geq n_2$, such that

$$\|\phi_{n_2 m}(q'_{n_2}) - \phi_{n_2 m}(q_{n_2})\| \leq 2\varepsilon \ . \tag{3.2}$$

Again, we introduce the following notation:

$$q'_m = \phi_{n_2 m}(q'_{n_2}) \ , \quad q_m = \phi_{n_2 m}(q_{n_2}) \quad \text{and} \quad p_m = \phi_{n_2 m}(p_{n_2}) \ .$$

As before, it holds that $p = \Phi_m(p_m)$ and $q_m = \Phi_m(q_m)$.

By construction, it already holds that $q'_m \sim_u p_m$. Choosing $\varepsilon < \frac{1}{2}$, (3.1) and (3.2) imply that

$$\|q'_m - q_m\| \leq 1 \ .$$

Then, from corollary 3.3.29 it follows that $q'_m \sim_u q_m$. Thus:

$$q_m \sim_u q'_m \sim_u p_m \quad \Rightarrow \quad q_m \sim_u p_m \ .$$

But then:

$$\begin{aligned}\Psi_{n_0}([P_j(q_{n_0})]) &= \Psi_m \circ (\phi_{n_0 m})_*([P_j(q_{n_0})]) = \Psi_m([P_j(\phi_{n_0 m}(q_{n_0}))]) \\ &= \Psi_m([P_j(q_m)]) = \Psi_m([P_j(p_m)]) = \ldots = \Psi_{n_0}([P_j(p_{n_0})]) \ .\end{aligned}$$

Yet, this shows injectivity. □

Remark 3.4.14.
For a direct system of proper C^*-algebras, the third claim can be extended to a Banach algebra direct limit with lemma 3.3.44. The direct limit in the category of Banach algebras can be constructed form the normed direct limit by metric completion.

3.4.2 The Grothendieck construction of K_{00}

Lemma and definition 3.4.15.
Let H be an abelian semi group[3]*. Define the equivalence relation $\sim$ on $H \times H$ by $(x_1, y_1) \sim (x_2, y_2)$, if and only if there is a $z \in H$, such that*

$$x_1 + y_2 + z = x_2 + y_1 + z \ .$$

Define $K(H) := {}^{(H\times H)}/_{\sim}$, then $K(H)$ is an abelian group, called ***Grothendieck group****, with the following operation:*

$$[(x_1, y_1)] + [(x_2, y_2)] := [(x_1 + x_2, y_1 + y_2)] \ .$$

The elements $[(x, y)] \equiv [x, y]$ are also denoted by $x - y$.

Remark 3.4.16.
The notation $x - y$ is not ambiguous, as $-x$ has a priori no meaning in a semi group. In fact it is only a formal difference, similar to fractions $\frac{a}{b}$, which are only formal quotients on an algebraic level. Two fractions are called equal $\frac{a}{b} = \frac{c}{d}$, if $a \cdot d = c \cdot b$. In same way, we could call $x_1 - y_1 = x_2 - y_2$, if $x_1 + y_2 = x_2 + y_1$. However, because we are using semi groups, which are not necessarily right cancellative, the condition is loosened somewhat by adding a constant $z \in H$ on both sides.

Proof 3.4.17.
First we need to show that $\sim$ is a proper equivalence relation:

Reflexivity: Holds for all z, since already $x + y = x + y$.

Symmetry: Follows from the symmetry of $=$.

Transitivity: Let $(x_1, y_1) \sim (x_2, y_2)$ by z and $(x_2, y_2) \sim (x_3, y_3)$ by w, then:

$$x_1 + y_2 + z = x_2 + y_1 + z \quad \text{and} \quad x_2 + y_3 + w = x_3 + y_2 + w \ .$$

[2] Here we could even demand that $\Phi_{n_1}(u_{n_1}) = u$ because of corollary 3.1.12, which also holds for monoids, etc. Then, since the extension of Φ_{n_1} on $\widetilde{A}_{n_1}$ has to preserve the unit, it follows that $u_{n_1} \in U_j(A_{n_1})$.

[3] I.e. an abelian monoid that does not necessarily have a neutral element.

Hence, choosing $u = x_2 + y_2 + z + w$, it follows that

$$\begin{aligned} x_1 + y_3 + u &= (x_1 + y_2 + z) + (x_2 + y_3 + w) \\ &= (x_2 + y_1 + z) + (x_3 + y_2 + w) \\ &= x_3 + y_1 + (x_2 + y_2 + z + w) \\ &= x_3 + y_1 + u \ , \end{aligned}$$

$$\Rightarrow \qquad (x_1, y_1) \sim (x_3, y_3) \ .$$

Next we need to show that $K(H)$ is a proper group. The first step is to show, that the operation is well defined. So let $(x_1', y_1') \in [(x_1, y_1)]$ and $(x_2', y_2') \in [(x_2, y_2)]$. Then there are z_1 and z_2, such that

$$x_1 + y_1' + z_1 = x_1' + y_1 + z_1 \quad \text{and} \quad x_2 + y_2' + z_2 = x_2' + y_2 + z_2 \ ,$$

$$\begin{aligned} \Rightarrow \quad x_1' + x_2' + y_1 + y_2 + z_1 + z_2 &= (x_1' + y_1 + z) + (x_2' + y_2 + z_2) \\ &= (x_1 + y_1' + z) + (x_2 + y_2' + z_2) \\ &= x_1 + x_2 + y_1' + y_2' + z_1 + z_2 \ . \end{aligned}$$

This shows that the operation is independent of the representative. Associativity follows from the associativity of the semi group. The neutral element is $e = [(a, a)]$ for all $a \in H$. Indeed, $(a, a) \sim (b, b)$ and

$$x + a + y = y + a + x \quad \Leftrightarrow \quad (x, y) \sim (x + a, y + a)$$

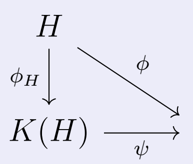
$$\Leftrightarrow \quad [x, y] + [a, a] = [x + a, y + a] = [x, y] \ .$$

Finally, the inverse element is given by $-[x, y] = [y, x]$:

$$[x, y] + [y, x] = [x + y, x + y] = [a, a] = e \ .$$

$\square$

Lemma 3.4.18 (**Grothendieck group universl property**).
The Grothendieck group satisfies the following universal property: There is a semi group morphism $\phi_H \colon H \to K(H)$, such that for all abelian groups G with semi group morphism $\phi \colon H \to G$, there is a unique group morphism $\psi \colon K(H) \to G$, such that the following diagram commutes:



Proof 3.4.19.
We choose $\phi_H\colon H \to K(H)$ by defining $\phi(a) = [a+a, a]$. It holds that

$$a + (b+b+a) = b + (a+a+b) \quad \Leftrightarrow \quad (a,b) \sim (a+a+b, b+b+a) \ ,$$

$$\begin{aligned} \Rightarrow \quad [a,b] &= [a+a+b, b+b+a] = [a+a+, a] + [b, b+b] \\ &= [a+a+, a] - [b+b, b] = \phi_H(a) - \phi_H(b) \ . \end{aligned}$$

Let now $\psi\colon K(H) \to G$ be a group morphism, then:

$$\psi([a,b]) = \psi(\phi_H(a) - \phi_H(b)) = \psi(\phi_H(a)) - \psi(\phi_H(b)) \ .$$

If this group morphism is to make the diagram commute, it has to holds that

$$\psi([a,b]) = \psi(\phi_H(a) - \phi_H(b)) = \psi(\phi_H(a)) - \psi(\phi_H(b)) \overset{!}{=} \phi(a) - \phi(b) \ .$$

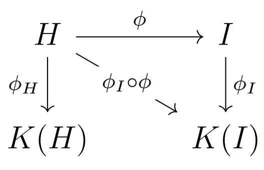
The right hand side defines such a morphism, proving existence. On the other hand, this equation also shows uniqueness, since

$$\psi([a,b]) - \psi'([a,b]) = \phi(a) - \phi(b) - (\phi(a) - \phi(b)) = 0 \ .$$

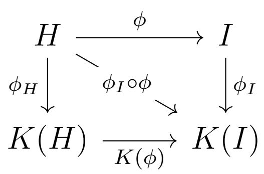
□

> **Theorem 3.4.20.**
> *The Grothendieck group defines a functor K from the category of abelian semi groups to the category of abelian groups.*

Proof 3.4.21.
To be a functor, we have to construct a group morphism $K(\phi)\colon K(H) \to K(I)$ from a semi group morphism $\phi\colon H \to I$. Consider the following diagram:



Then, $\phi_I \circ \phi\colon H \to K(I)$ is a semi group morphism to an abelian group $K(I)$. By the universal property of the Grothendieck group, there is a unique morphism $K(\phi)$, such that $K(\phi) \circ \phi_H = \phi_I \circ \phi$, i.e.



□

Remark 3.4.22.
In the proof of lemma 3.4.18 we have also constructed the unique morphism. By comparing the diagrams, we find

$$\begin{aligned} K(\phi)([a,b]) &= \phi_I(\phi(a)) - \phi_I(\phi(b)) \\ &= [\phi(a)+\phi(a),\phi(a)] - [\phi(b)+\phi(b),\phi(b)] \\ &= [\phi(a)+\phi(a),\phi(a)] + [\phi(b),\phi(b)+\phi(b)] \\ &= [\phi(a)+\phi(a)+\phi(b),\phi(a)+\phi(b)+\phi(b)] \\ &= [\phi(a),\phi(b)]\ . \end{aligned}$$

Theorem 3.4.23.
The Grothendieck group is additive, i.e.

$$K(A\oplus B) \cong K(A)\oplus K(B)\ .$$

Proof 3.4.24.
By construction, we need to show that

$$^{A\oplus B\times A\oplus B}/_{\sim} \cong {}^{A\times A}/_{\sim} \oplus {}^{B\times B}/_{\sim}\ .$$

We define a group morphism by

$$\phi\colon [(a_1,b_1),(a_2,b_2)] \longmapsto ([a_1,a_2],[b_1,b_2])\ .$$

First, we need to show, that this map is well defined, i.e. independent of the representative. So let $(a_1',b_1',a_2',b_2') \in (a_1,b_1)-(a_2,b_2)$, i.e. it holds that

$$(a_1',b_1')+(a_2,b_2)+(z,w) = (a_1,b_1)+(a_2',b_2')+(z,w)\ .$$

With the natural operation on the direct sum, this reads:

$$(a_1'+a_2+z, b_1'+b_2+w) = (a_1+a_2'+z, b_1+b_2'+w)$$

$$\Leftrightarrow \quad \begin{aligned} a_1'+a_2+z &= a_1+a_2'+z \\ b_1'+b_2+w &= b_1+b_2'+w \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} (a_1',a_2') &\sim (a_1,a_2) \\ (b_1',b_2') &\sim (b_1,b_2) \end{aligned}\ .$$

$$\Leftrightarrow \quad ((a_1',a_2'),(b_1',b_2')) \in ([a_1,a_2],[b_1,b_2])\ .$$

This shows that the map does not depend on the representative. Direct calculation shows, that ϕ is a proper group morphism. Surjectivity is immediate by the definition. For injectivity, we have to show that $\mathrm{Ker}(\phi) = e$. Assume that $\phi([(a_1,b_1),(a_2,b_2)]) = (e,e)$ then it has to hold that

$$(a_1,a_2)\in[x,x] \quad \text{and} \quad (b_1,b_2)\in[y,y]\ ,$$

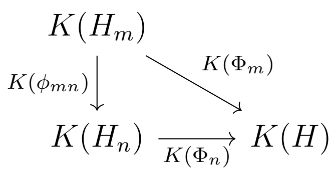
for $x\in A$ and $y\in B$. This implies that $a_1=a_2$ and $b_1=b_2$. But $[(a_1,a_1),(b_1,b_1)]$ is the unit element of $^{A\oplus B\times A\oplus B}/_{\sim}$. □

Theorem 3.4.25.
The functor K commutes with direct limits over $\mathbb{N}$.

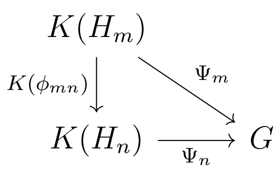
Proof 3.4.26.
Let (H_m, ϕ_{mn}) be a direct system of abelian semi groups and (H, Φ_n) be the direct limit. By functoriality of K, $(K(H_m), K(\phi_{mn}))$ is a direct system of abelian groups. From functoriality it also follows that (cf. the proof of theorem 3.4.12)

$$K(\Phi_n) \circ K(\phi_{mn}) = K(\Phi_n \circ \phi_{mn}) = K(\Phi_m) \ ,$$

so the following diagram commutes:

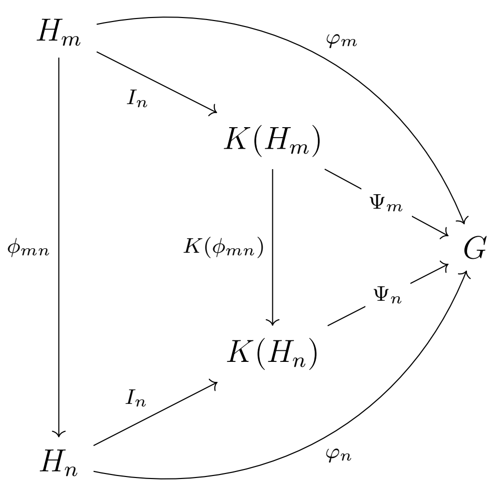


Assume, that (G, Ψ_n) also satisfies the mapping property



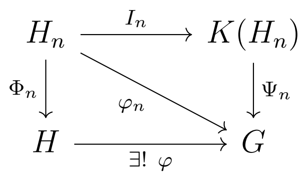
$$\Rightarrow \quad \Psi_m = \Psi_n \circ K(\phi_{mn})$$

where G is a group and Ψ_n are group morphisms. Let $I_n \equiv \phi_{H_n} \colon H_n \longrightarrow K(H_n)$ be the canonical morphism from the universal property of the Grothendieck group. Let $\varphi_n := \Psi_n \circ I_n$, and consider the following diagram (the left part of the diagram commutes, since it is the definiton of $K(\phi_{mn})$, as can be seen in proof 3.4.21):



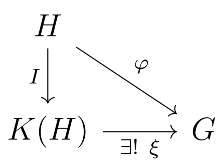
$$\Rightarrow \quad \varphi_m = \varphi_n \circ \phi_{mn} \ .$$

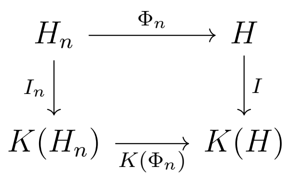
Because of the universal property of the direct limit (lower triangle), there is a unique group morphism $\varphi\colon H \longrightarrow G$, such that the following diagram commutes:



$$\Rightarrow \quad \varphi \circ \Phi_n = \varphi_n \ .$$

Let $I = \phi_H\colon H \longrightarrow K(H)$. Again, form the unique property, it follows that there is a unique $\xi\colon K(H) \longrightarrow G$, such that the left diagram commutes. The right diagram is again, just the definition of $K(\Phi_n)$



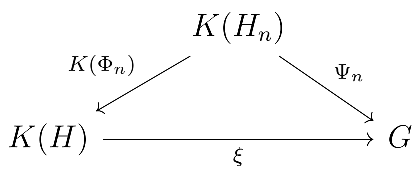


From the right diagram, we see that $K(\Phi_n) \circ I_n = I \circ \Phi_n$. We find:

$$\xi \circ K(\Phi_n) \circ I_n = \xi \circ I \circ \Phi_n \equiv \varphi \circ \Phi_n = \varphi_n \equiv \Psi_n \circ I_n$$

$$\Rightarrow \quad \xi \circ K(\Phi_n) = \Psi_n \ .$$

Hence, the following diagram commutes:



To show, that $K(H)$ is the direct limit, it remains to show that ξ is the unique group morphism with that property. So assume, that $\xi'\colon K(H) \to G$ also satisfies $\xi' \circ K(\Phi_n) = \Psi_n$ for all $n \in \mathbb{N}$. Then:

$$\xi' \circ I \circ \Phi_n = \xi' \circ K(\Phi_n) \circ I_n = \Psi_n \circ I_n = \varphi_n = \varphi \circ \Phi_n$$

$$\Rightarrow \quad \xi' \circ I = \varphi = \xi \circ I \quad \Rightarrow \quad \xi' = \xi \ .$$

□

Definition 3.4.27.
The ***K_{00}* group** $K_{00}(A)$ is defined by

$$K_{00}(A) \coloneqq K(V(A)) \ .$$

Since V and K are functors,

$$K_{00} = K \circ V$$

is also a functor, from local C^*-algebras to abelian groups. The induced group morphism $K_{00}(\phi)$ will also be denoted by ϕ_*.

Example 3.4.28.
Consider $M_n(\mathbb{C})$. Up to similarity, the orthogonal projections are given by $I_k = \mathrm{diag}(1,\dots,1,0,\dots,0)$, where k denotes the number of ones. Hence the equivalence classes of ${}^{M_n(\mathbb{C})}/_{\sim_s}$ are $[I_k] = [k]$ for $k = 0,\dots,n$. Let K denote a sequence of length n, with k ones and $n-k$ zeroes and denote $I_K = \mathrm{diag}(K)$. It holds that $I_k \sim_s I_K$, such that $I_K \in [k]$.
In case of $M_\infty(\mathbb{C})$, one can choose K and L, such that $I_K \perp I_L$ for $I_K \in [k]$ and $I_L \in [\ell]$. It holds that

$$I_K + I_L = I_{K+L} \in [j+\ell] \quad \Rightarrow \quad [j]+[\ell] = [j+\ell] \ .$$

Hence, with remark 3.4.4 it follows that

$$V(\mathbb{C}) = V(M_n(\mathbb{C})) = {}^{M_\infty(\mathbb{C})}/_{\sim_s} = \mathbb{N}_0 \ .$$

Thus, since the Grothendieck group of $\mathbb{N}_0$ is by construction $\mathbb{Z}$, it follows that

$$K_{00}(\mathbb{C}) = K(V(\mathbb{C})) = K(\mathbb{N}_0) = \mathbb{Z} \ .$$

3.4.3 Construction of the K_0-group

Let A be a local C^*-algebra. We define $A^+ := A \times \mathbb{C}$ with the operations

$$(a,z)+(b,w) = (a+b, z+w) \ , \qquad (a,z)(b,w) = (ab+wa+zb, zw)$$
$$\text{and} \quad (a,z)^* = (a^*, \overline{z}) \ .$$

A $*$-morphism $\phi\colon A \longrightarrow B$ induces a $*$-morphism $\phi^+\colon A^+ \longrightarrow B^+$ by

$$(a,z) \longmapsto (\phi(a), z) \ .$$

Lemma 3.4.29.
Let (A_n, ϕ_{mn}) be a direct system of local C^*-algebras with algebraically direct limit (A, Φ_n). Then (A^+, Φ_n^+) is the direct limit of (A_n^+, ϕ_{mn}^+), where

$$\phi_{mn}^+((a_m,z)) = (\phi_{mn}(a_m), z) \quad \text{and} \quad \Phi_n^+((a_n,z)) = (\Phi_n(a_n), z) \ .$$

Proof 3.4.30.
We need to show, that $\Phi_n^+ \circ \phi_{mn}^+ = \Phi_m^+$. Let $(a_m, z) \in A_m^+$, then it holds that

$$\Phi_m^+((a_m,z)) = (\Phi_m(a_m), z) = ((\Phi_n \circ \phi_{mn})(a_m), z) = (\Phi_n^+ \circ \phi_{mn}^+)((a_m,z)$$
$$\Rightarrow \quad \Phi_n^+ \circ \phi_{mn}^+ = \Phi_m^+ \ .$$

In the same way, it can be seen, that (A_m^+, ϕ_{mn}^+) is a direct system.
Consider the direct limit $(\bigsqcup_i {}^{A_i^+}/\sim, \pi_n)$ from theorem 3.1.10. By definition, there is a unique morphism $\xi\colon \bigsqcup_i {}^{A_i^+}/\sim \to A^+$, such that the following diagram commutes:

$$\begin{array}{ccc} A_n^+ & \xrightarrow{\quad\pi_n\quad} & \bigsqcup_i A_i^+/\sim \\ & \underset{\Phi_n^+}{\searrow} \quad \quad \underset{\xi}{\swarrow} & \\ & A^+ & \end{array}$$

Furthermore, in the proof of theorem 3.1.10, the map ξ is explicitly constructed. Here it reads:

$$\xi([(a_n, z); n]) = \Phi_n^+(a_n, z) = (\Phi_n(a_n), z) \ .$$

To see that ξ is well defined, let $(b_m, w) \in [(a_n, z); n]$, then by definition of $\sim$, there is a $\mathbb{N} \ni k \geq m, n$, such that

$$\phi_{nk}(a_n, z) = (\phi_{nk}(a_n), z) = (\phi_{mk}(b_m), w) = \phi_{mk}^+(b_m, w)$$

$$\Rightarrow \quad z = w \quad \text{and} \quad \phi_{nk}(a_n) = \phi_{mk}(b_m) \ .$$

$$\begin{aligned} \Rightarrow \quad \xi([(b_m, w), m]) &= (\Phi_m(b_m), w) = (\Phi_k(\phi_{mk}(a)), w) \\ &= (\Phi_k(\phi_{nk}(a_n)), z) = (\Phi_n(a_n), z) \\ &= \xi([(a_n, z); n]) \ . \end{aligned}$$

This shows that ξ is indeed well defined. It remains to show that ξ is bijective. **Surjectivity:** Let $(a, z) \in A^+$, then there is an $a_n \in A_n$, such that $a = \Phi_n$. It follows that:

$$\Phi_n^+((a_n, z)) = (\Phi_n(a_n), z) = (a, z) \ .$$

But then:

$$(a, z) = \Phi_n^+(a_n, z) = (\xi \circ \pi_n)((a_n, z)) = \xi([(a_n, z); n]) \ .$$

Injektivity: Let $[(a_n, z); n], [(b_m, w); m] \in \bigsqcup_i A_i^+/\sim$ and assume that $\xi([(a_n, z); n]) = \xi([(b_m, w); m])$. Then:

$$\xi([(a_n, z); n]) = (\Phi_n(a_n), z) = (\Phi_m(b_m), w) = \xi([(b_m, w); m])$$

$$\Rightarrow \quad z = w \quad \text{and} \quad \Phi_n(a_n) = \Phi_m(b_n) \ .$$

By definition of A^+ as $\bigsqcup_i A_i/\sim \times \mathbb{C}$, the Φ_n are actually π_n. Hence:

$$\Phi_n(a_n) = \Phi_m(b_n) \quad \Leftrightarrow \quad \pi_n(a_n) = \pi_m(b_m) \quad \Leftrightarrow \quad a_n \sim b_m \ .$$

Thus, there is a $\mathbb{N} \ni k \geq m, n$, such that $\phi_{nk}(a_n) = \phi_{mk}(b_m)$. Yet, together with $w = z$, this means $[(a_n, z); n] = [(b_m, w); m]$ □

Lemma 3.4.31.
If and only if A is non-unital, then A^+ and $\widetilde{A}$ are $*$-isomorphic. In the unital case it holds that $A^+ \cong A \oplus \mathbb{C}$, where the operations of $A \oplus \mathbb{C}$ are component wise. It especially follows that A^+ is a local C^*-algebra in the non-unital case.

Proof 3.4.32.

i) In the non-unital case, we can write $\widetilde{A} = \{\pi(a) + z\mathbb{1} \mid a \in A, z \in \mathbb{C}\}$, where we used the uitalization from lemma 2.1.7. Then, $\phi\colon (a, z) \mapsto \pi(a) + z\mathbb{1} \in \widetilde{A}$ is a $*$-morphism, which can be seen by direct calculation. Furthermore, ϕ is surjective.

For injectivity we need to show that $\mathrm{Ker}(\phi) = 0$. Assume that $\phi((a, z)) = 0$ for $(a, z) \neq 0$. This implies that $0 = \pi(a) + z\mathbb{1}$, and thus $\pi(a) \sim \mathbb{1}$. Since A is non-unital, there is no $a \in A$, such that $\pi(A) \sim \mathbb{1}$. Hence $\mathrm{Ker}(\phi) = 0$.

For the opposite direction, we can also show the equivalent statement: "If A is unital, then $A^+ \ncong \widetilde{A}$". It is enough to show that $\mathrm{Ker}(\phi) \neq 0$. Choosing $\mathbf{1} \in A$, it follows that

$$\phi((\mathbf{1}, 1)) = \pi(\mathbf{1}) - \mathbb{1} = \mathbb{1} - \mathbb{1} = 0 \quad \Rightarrow \quad \mathrm{Ker}(\phi) \neq 0 \ .$$

ii) Consider the map $\psi\colon A^+ \longrightarrow A \oplus \mathbb{C}\ ,\ (a, z) \longmapsto [a + z\mathbf{1}; z]$. The addition and $*$-map act component wise. To see that ψ is a $*$-morphism, all that is left, is to check the morphism property for the multiplication:

$$\begin{aligned}\psi((a, z)(b, w)) = \psi((ab + wa + zb, zw)) &= [ab + wa + zb + zw\mathbf{1}, zw] \\ &= [a + z\mathbf{1}, z][b + w\mathbf{1}, w] = \psi((a, z))\psi((b, w)) \ .\end{aligned}$$

Surjectivity follows from $\psi((a - z\mathbf{1}, z)) = [a - z\mathbf{1} + z\mathbf{1}, z] = [a, z]$. For injectivity, assume that $[0, 0] = \psi((a, z))$. Then:

$$[0, 0] = \psi((a, z)) = [a + z\mathbf{1}, z] \quad \Rightarrow \quad z = 0 \quad \Rightarrow \quad [0, 0] = [a, 0]$$

$$\Rightarrow \quad \begin{matrix} a = 0 \\ z = 0 \end{matrix} \quad \Rightarrow \quad \mathrm{Ker}(\psi) = 0 \ .$$

Thus ψ is an isomorphism.

□

Remark 3.4.33 (Notation).
Formally, we would need to write $[a, z] \equiv a \oplus z \in A \oplus \mathbb{C}$ and $(a, z) \in A^+$. The different brackets are used to differentiate the different multiplication structures of $A \oplus \mathbb{C}$ and A^+. In the latter case, the product behaves as the distributive law dictates, such that

$$(a, z)(b, w) := (ab + zb + wa, zw) \ \leftrightarrow \ ab + zb + wa + zw = (a + z)(b + w) \ .$$

Hence, one sometimes identifies (a, z) with $a + z$. This can be misleading, as $a + z \neq a + z\mathbf{1}$, which is however also a common notation.
On the other hand, this is not possible for $A \oplus \mathbb{C}$, since $(a \oplus z)(b \oplus w) = ab \oplus zw$. This is also the reason, why we could not use the natural map $(a, z) \mapsto [a, z]$ to show isomorphy in the unital case.

Assume now, that A is unital and consider the map

$$\phi\colon A^+ \longrightarrow A \oplus \mathbb{C} \;, \quad (a,z) \longmapsto (a + z\mathbf{1}) \oplus z \;.$$

Similar to the non-unital case, this is a $*$-morphism. So A^* is also a local C^*-algebra in the non-unital case, where the $*$-norm on $A \oplus \mathbb{C}$ is

$$\|a \oplus z\| := \max(\|a\|, |z|) \;.$$

We observe that ${}^{A^+}\!/\!_A = \mathbb{C}$ and by example 3.4.28 we find

$$K_{00}({}^{A^+}\!/\!_A) = \mathbb{Z} \;.$$

Let $\pi\colon A^+ \to {}^{A^+}\!/\!_A = \mathbb{C}$ be the canonical projection (quotient map) on the second component. Although not a projection in the C^*-algebra sense, it still is a $*$-morphism.

Definition 3.4.34.
With $\pi_* \equiv K_{00}(\pi)$, we define the $\boldsymbol{K_0}$**-group** as:

$$K_0(A) := \mathrm{Ker}(\pi_*) \subseteq K_{00}(A^+) \;.$$

The induced map π_* is a map $K_{00}(A^+) \longrightarrow K_{00}(\mathbb{C}) = \mathbb{Z}$. Since π_* is a morphism of abelian groups, the kernel is a normal sub group. Hence $K_0(A) \subset K_{00}(A^+)$ is an abelian group.

Definition and lemma 3.4.35.
Let $\phi\colon A \to B$ be a $*$-morphism between local C^*-algebras. With

$$K_0(\phi) := K_{00}(\phi^+)\colon K_{00}(A^+) \longrightarrow K_{00}(B^+) \;,$$

K_0 becomes a functor from local C^*-algebras to abelian groups.

As usual, if the context is understood, $K_0(\phi)$ is denoted by ϕ_*.

3.4.4 Properties of K_0 and K_{00}

Notation 3.4.36.
Let p_n denote the matrix that is diagonal with n- units:

$$p_n := \mathrm{diag}(\underbrace{1,\ldots,1}_{n\text{-times}},0,\ldots,0) \;.$$

The notation p is inspired by the fact, that p_n is a projection.

Lemma 3.4.37.
Let $p \in \mathrm{Proj}(M_k(A^+))$ and $p_n \in M_k(\mathbb{C})$. If $\pi(p) \sim p_n$, then there is a p' with $p' \sim_u p$, such that

$$p' - p_n \quad \in M_k(A) \;.$$

Proof 3.4.38.
Since $M_k(\mathbb{C}) = \mathcal{L}(\mathbb{C}^k)$ and $\mathbb{C}^k$ is a finite dimensional vector space, the projections are given by p_n, up to similarity equivalence (and equivalently by unitary equivalence). Hence, from $\pi(p) \sim p_n$ it follows that there is a $u \in U_k(\mathbb{C}) =: U(k)$, such that $u^*\pi(p)u = p_n$. With $z \in \mathbb{C}$ as $(0, z) \in A^+$, we can consider $u \in U_k(A^+)$ and $p_n \in M_k(A^+)$. It follows that $\pi(u) = u$ and $\pi(p_n) = p_n$. Define $p' = upu^*$, then:

$$\pi(p') = \pi(upu^*) = u\pi(p)u^* = p_n = \pi(p_n) \ .$$

Hence p' and p_n have the same second component for all coefficients. Understanding $(a, 0) \equiv a \in A$, it follows that

$$p' - p_n \quad \in M_k(A) \ .$$

□

Remark 3.4.39.
In the proof we have used the isomorphy $A^+ \supset A \times \{0\} \cong A$. This does not hold for $A \times \{c\}$ for $c \neq 0$. Hence $p' - p_n \in M_k(A)$ means that $\pi(p') = \pi(p_n)$. In that case one also writes

$$p' \equiv p_n \mod M_k(A) \quad \overset{\text{notation}}{\Longleftrightarrow} \quad p' - p_n \in M_k(A) \ .$$

Theorem 3.4.40 (**Standard picture of $K_0(A)$**).

i) The elements of $K_0(A)$ are the elements of $K_{00}(A^+)$ that have the form $[p] - [q]$ with $p, q \in \mathrm{Proj}(M_k(A^+))$ and $p - q \in M_k(A)$.

ii) The elements of $K_0(A)$ have the form $[p] - [p_n]$ where $p \in \mathrm{Proj}(M_k(A^+))$ and $p_n = \mathrm{diag}(1, \dots, 1, 0, \dots, 0) \in M_k(A^+)$, such that $p - p_n \in M_k(A)$.

iii) If $[p] - [q] = 0$ in $K_0(A)$, where p, q are chosen as in (i), then there is an $m \in \mathbb{N}$, such that

$$\mathrm{diag}(p, p_m) \sim \mathrm{diag}(q, p_m) \quad \text{in } M_{k+m}(A^+) \ .$$

Furthermore, there is an $n \geq m$, such that one can exchange the relation $\sim$ by $\sim_h$ or $\sim_u$ in $M_{k+n}(A^+)$.

Remark 3.4.41.
We are somewhat sloppy in the notation here. With $[p] - [q] \in K_{00}(A^+)$ we mean $[\Phi_k(p)] - [\Phi_k(q)]$, etc. Furthermore $p_m \in M_m(A^+)$ is equal to $\mathbb{1}$.

Proof 3.4.42.

i) Let $p, q \in \mathrm{Proj}(M_k(A^+))$ with $p - q \in M_k(A)$, then $\pi(p) = \pi(q)$. It follows that $\pi_*([p] - [q]) = 0$, i.e. $[p] - [q] \in \mathrm{Ker}(\pi_*)$.

On the other hand let $[p] - [q] \in K_0$, with $p, q \in \mathrm{Proj}(M_k(A^+))$ for some $k \in \mathbb{N}$. Then, by definition of $K_0(A)$, it holds that

$$0 = \pi_*([p] - [q]) = [\pi(p)] - [\pi(q)] \quad \text{in } K_{00}(\mathbb{C}) = \mathbb{Z} \ .$$

By construction of $K_{00} = K \circ V$, the elements $[\pi(p)]$ and $[\pi(q)]$ are elements of $V(\mathbb{C}) = \mathbb{N}$. By definition, the last equation reads formally

$$\Leftrightarrow \quad \exists\, [z] \in V(\mathbb{C})\colon [\pi(q)] + [z] = [\pi(p)] + [z] \ .$$

Since $\mathbb{N}$ is a cancellative monoid, the equation implies $[\pi(p)] = [\pi(q)]$. These are equivalence classes of projections in $M_K(\mathbb{C})$. Hence there is a $p_n \in M_k(\mathbb{C})$, such that $\pi(p) \sim \pi(q) \sim p_n$.

From lemma 3.4.37 it follows that there are $p' \sim_u p$ and $q' \sim q$, such that

$$p' - p_n \in M_k(A) \quad \text{and} \quad q' - p_n \in M_k(A)$$

$$\Rightarrow \quad p' - q' \in M_k(A) \ .$$

With $[p] = [p']$ and $[q] = [q']$ we find

$$[p] - [q] = [p'] - [q'] \ .$$

Yet, p' and q' satisfy the demanded conditions.

ii) Let $[p] - [q]$ be in $K_{00}(A^+)$, with p, q as in part (i). For $n \geq k$, p_n is the unit matrix and thus $p_n q = q$. From lemma 3.3.21 it follows that $p_n - q$ is a projection. In $M_\infty(A^+)$, we can move p along the diagonal (by conjugation with unitary matrices), such that we obtain a p' with

$$p' \sim_u p \quad \text{and} \quad p' \perp p_n \ .$$

With $p'q = p'p_n q = 0q = 0$, one sees that $p' \perp p_n - q$. The same holds for q, such that

$$[p_n - q] + [q] = [p_n] \ .$$

In $K_{00}(A^+)$ we calculate:

$$[p' + p_n - q] - [p_n] = ([p'] + [p_n - q]) - [p_n] = [p'] - [q] = [p] - [q] \ .$$

iii) Since $0 = [0] = [p] - [q]$, there is an $r \in \mathrm{Proj}(M_m(A^+))$, such that

$$[p] + [r] = [q] + [r] \quad \text{in } V(A^+) \ .$$

In $M_m(A^+)$, p_m is the unit matrix, such that $r \leq p_m$. With the same methods as in (ii), we calculate:

$$\begin{aligned}[\mathrm{diag}(p, p_m)] &= [p] + [p_m] = [p] + [r] + [p_m - r] \\ &= [q] + [r] + [p_m + r] = [\mathrm{diag}(q, p_m)] \ .\end{aligned}$$

But this means that $\mathrm{diag}(p, p_m) \sim \mathrm{diag}(q, p_m)$. Choosing $n \geq m$ large enough (such that one obtains a matrix with matrices as coefficient), the relations become equivalent.

□

Theorem 3.4.43.
Both functors K_0 and K_{00} are homotopy invariant, additive and commute with direct limits over $\mathbb{N}$.

Proof 3.4.44.

i) From theorem 3.4.12 it already follows that $V(\phi_0) = V(\phi_1)$ for homotopic maps ϕ_0 and ϕ_1. Hence:

$$K_{00}(\phi_0) = K(V(\phi_0)) = K(V(\phi_1)) = K_{00}(\phi_1) \ .$$

Furthermore, let $\Psi_t \colon [0,1] \to B$ be continuous path of $*$-morphisms between the maps $\phi_0, \phi_1 \colon A \to B$. Then $\Psi_t^+ \colon [0,1] \to B^+$ is a continuous path of $*$-morphisms between $\phi_0^+, \phi_1^+ \colon A^+ \to B^+$. By definition, $K_0(\phi) = K_{00}(\phi^+)$, which shows that $K_0(\phi_0) = K_0(\phi_1)$, by the first part.

ii) Again, by theorem 3.4.12 and 3.4.23, the functors V and K are already additive. Hence K_{00} is additive.

The maps π and π_* act component wise on the elements of the direct sum. Let $[p,a] - [q,b] \in K_0(A \oplus B)$. This means (since $\mathbb{N}$ is cancellative) that

$$0 = \pi_*([p,a] - [q,b]) = \pi_*[p,a] - \pi_*[q,b] = [\pi(p), \pi(a)] - [\pi(q), \pi(b)] \ ,$$

$$\Leftrightarrow \quad \pi(p) = \pi(q) \quad \text{and} \quad \pi(a) = \pi(b) \ .$$

On the other hand, for $([p]-[q],[a]-[b]) \in K_0(A) \oplus K_0(B)$, the same condition applies:

$$(0,0) = \pi_*([p] - [q], [a] - [b]) = ([\pi(p)] - [\pi(q)], [\pi(a)] - [\pi(b)]) \ ,$$

$$\Leftrightarrow \quad \pi(p) = \pi(q) \quad \text{and} \quad \pi(a) = \pi(b) \ .$$

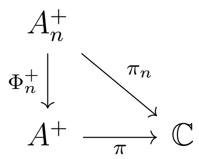
This shows that

$$[p,a] - [q,b] \longmapsto ([p] - [q], [a] - [b])$$

defines an isomorphism $K_0(A \oplus B) \longrightarrow K_0(A) \oplus K_0(B)$.

iii) Since V and K commute with direct limit over $\mathbb{N}$ (see theorem 3.4.12 and 3.4.25), so does K_{00}.

Let (A_n, ϕ_{mn}) be a direct system of local C^*-algebras with direct limit (A, Φ_n). As seen in lemma 3.4.29, (A^+, Φ_n^+) is the direct limit of (A_n^+, ϕ_{mn}^+). Let $\pi_n\colon A^+ \to \mathbb{C}$ and $\pi\colon A^+ \to \mathbb{C}$ be the projections on the second component, then it holds that



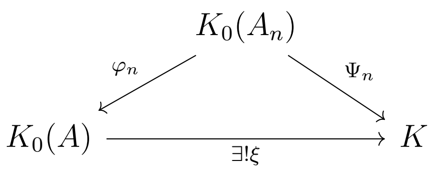
$$\Rightarrow \quad \pi \circ \Phi_n^+ = \pi_n$$

and by functoriality of K_{00} it also follows:

$$K_{00}(\pi) \circ K_{00}(\Phi_n^+) = K_{00}(\pi) \ .$$

Thus, the kernel of $K_{00}(\pi_n)$ is mapped onto the kernel of $K_{00}(\pi)$ by the map $K_{00}(\Phi_n^+)\colon K_{00}(A_n^+) \to K_{00}(A)$. The restriction defines a morphism: $\varphi_n\colon K_0(A_n) \to K_0(A)$. Furthermore, it holds that $\varphi_n \circ K_{00}(\phi_{mn}^+) = \varphi_m$.

Let (K, Ψ_n) be the inductive limit, i.e. $K := \lim\limits_{\rightarrow} K_0(A_n)$, then there is a unique morphism $\xi\colon K \to K_0(A)$, such that



i.e. $\xi \circ \Psi_n = \varphi_n$. It remains to show, that ξ is an bijective.

Surjectivity: Let $[p] - [q] \in K_0(A)$, then there are an $n \in \mathbb{N}$ and $p_n, q_n \in \operatorname{Proj}(M_\infty(A_n^+))$, such that $p = \Phi_n^+(p_n)$ and $q = \Phi_n^+(q_n)$. It follows that:

$$\begin{aligned} 0 = K_{00}(\pi)([p] - [q]) &= [\pi(p)] - [\pi(q)] = [(\pi \circ \Phi_n^+)(p_n)] - [(\pi \circ \Phi_n^+)(q_n)] \\ &= [\pi_n(p_n)] - [\pi_n(q_n)] = K_{00}(\pi_n)([p_n] - [q_n]) \ , \end{aligned}$$

such that $[p_n] - [q_n] \in K_0(A_n)$. Hence,

$$\begin{aligned} \xi(\Psi_n([p_n] - [q_n])) &= \varphi_n([p_n] - [q_n]) = K_{00}(\Phi_n^+)([p_n] - [q_n]) \\ &= [\Phi_n^+(p_n)] - [\Phi_n^+(q_n)] = [p] - [q] \ , \end{aligned}$$

which shows surjectivity.

Injectivity: Let $x \in K$, such that $\xi(x) = 0$. There are an $n \in \mathbb{N}$ and $x_n \in K_{00}(A_n^+)$, such that $x = \Psi_n(x_n)$. It holds that

$$0 = \xi(x) = \xi(\Psi_n(x_n)) = \varphi_n(x_n) = K_{00}(\Phi_n^+)(x_n) \in K_{00}(A^+) \ ,$$

i.e. $x_n \sim 0$. Since $K_{00}(A^+)$ is the direct limit of $K_{00}(A^+_m)$, as proven above, there is an $m \geq n$, such that

$$x_m := K_{00}(\phi^+_{nm})(x_n) = 0$$

$$\Rightarrow \quad x = \Psi_n(x_n) = \Psi_m(K_{00}(\phi^+_{nm})(x_m)) = \Psi_m(x_m) = \Psi_m(0) = 0 \ .$$

This shows, that $\mathrm{Ker}(\xi) = 0$, which is equivalent to ξ being injective.

□

Corollary 3.4.45.
It holds that $K_{00}(A) \cong K_{00}(A \otimes \mathcal{K})$ and $K_0(A) \cong K_0(A \otimes \mathcal{K})$.

Proof 3.4.46.
The argument is the same for both K_{00} and K_0, so let F denote either of the functors. Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle \phi_{1m}}\downarrow & & \downarrow{\scriptstyle \phi_{1n}} \\ M_m(A) & \xrightarrow[\phi_{mn}]{} & M_n(A) \end{array}$$

Here we used the constant system (A, Id_A) from example 3.1.14 with direct limit $A_\infty = A$. From lemma 3.2.18 we now that$M_\infty(A) \cong M_\infty(M_n(A))$. Since booth K_{00} and K_0 are constructed from M_∞ it follows that $F(\phi_{1m})$ and $F(\phi_{1n})$ are isomorphisms. Hence, applying the functor F to the diagram results in

$$\begin{array}{ccc} F(A) & \xrightarrow{\mathrm{Id}_{F(A)}} & F(A) \\ {\scriptstyle F(\phi_{1m})}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle F(\phi_{1n})} \\ F(M_m(A)) & \xrightarrow[\phi_{mn}]{} & F(M_n(A)) \end{array}$$

This means, that the direct limits of the direct systems $(F(A), \mathrm{Id}_{F(A)})$ and $(F(M_n(A)), F(\phi_{mn}))$ are isomorphic.

Since K_{00} and K_0 commute with direct limits by theorem 3.4.43, we find:

$$\begin{aligned} F(A) = F(\varinjlim A) \cong \varinjlim F(A) \cong \varinjlim F(M_n(A)) \\ \cong F(\varinjlim M_n(A)) = F(M_\infty(A)). \end{aligned}$$

In fact, for K_{00} and K_0 we consider $\mathrm{Idem}(M_\infty(A))/\sim_h$. Because of lemma 3.3.44, completion does not add new equivalence classes, such that

$$F(A) \cong F(M_\infty(A)) = F(\widehat{M_\infty(A)}) \equiv F(A \otimes \mathcal{K}) \ .$$

□

We consider the map

$$\varphi\colon V(A^+) \longrightarrow K_0(A)\ , \quad [p] \longmapsto [p]-[p_n]\ ,$$

where n is the rank of $\pi(p)$. Since $\pi(p)$ has the same rank as p_n, it holds (lemma 3.4.37) that there is a $p' \sim_u p$ with $p' - p_n \in M_k(A)$. Hence $[p]-[p_n] = [p'] - [p_n] \in K_0(A)$ by theorem 3.4.40.

Let $p \perp q$ with $\operatorname{rank}(\pi(q)) = m$, then $\pi(p) \perp \pi(q)$,since π is a $*$-morphism. So φ is a semi group morphism. Furthermore $\pi(p+q)$ has the rank $\operatorname{rank}(\pi(p)) + \operatorname{rank}(\pi(q))$, and thus φ is additive.

Corollary 3.4.47.
The map

$$V(A^+) \longrightarrow K_0(A)\ , \quad [p] \longmapsto [p]-[p_n]$$

induces a group morphism $\psi\colon K_{00}(A^+) \to K_0(A)$.

Proof 3.4.48.
Using the universal property of the Grothendieck group yields:

$$\begin{array}{ccc} V(A^+) & & \\ {\scriptstyle \Phi_{V(A^+)}}\downarrow & \searrow^{\varphi} & \\ K_{00}(A^+) & \xrightarrow[\psi]{} & K_0(A) \end{array}$$

In the proof of lemma 3.4.18 we have also constructed ψ explicitly. Let $n = \operatorname{rank}(\pi(p))$ and $m = \operatorname{rank}(\pi(q))$:

$$\begin{aligned} \psi([p]-[q]) &= \varphi([p]) - \varphi([q]) = ([p]-[p_n]) - ([q]-[p_m]) \\ &= [p]-[p_n] + ([p_m]-[q]) = [p]+[p_m]-[q]+[p_n] \end{aligned}$$

□

Composition with the canonical map $K_{00}(A) \to K_{00}(A^+)$ yields a map

$$\omega_A\colon K_{00}(A) \longrightarrow K_0(A)\ .$$

Definition 3.4.49.
A local C^*-algebra A is called **stablely unital**, if $M_\infty(A)$ has an approximate unit consisting of projections.

Remark 3.4.50.
Since the identity is always a projection, a unital C^*-algebra is always stablely unital. From lemma 3.3.44 it follows, that A is stablely unital, if and only if $A \otimes \mathcal{K}$ has an approximate unit of projections.

Theorem 3.4.51.
Let A be stablely unital, then $\omega_A\colon K_{00}(A) \to K_0(A)$ is an isomorphism of abelian groups. This especially holds, if A is unital.

Proof 3.4.52.
Under these assumptions $M_\infty(A)$ has a dense local C^*-sub-algebra, that is the algebraic direct limit of a direct system of unital C^*-algebras. (For example consider $pM_n(A)p$, where $p \in M_n(A)$ belongs to the approximate unit. In $pM_n(A)p$, the unit element is p itself. It is a result of functional analysis, that the limit is dense in $M_\infty(A)$.)

Since K_{00} and K_0 commute with direct limits, it is enough to show the statement for the unital case. So assume that A is unital. Then $A^+ \cong A \oplus \mathbb{C}$. Then, because of the additivity of K_{00}, it follows that $K_{00}(A^+) \cong K_{00}(A) \oplus \mathbb{Z}$.[4]Furthermore, it follows that $K_{00}(A) = \mathrm{Ker}(\pi_*)$. Hence $K_{00}(A^+) \cong K_0(A) \oplus \mathbb{Z}$ and $K_{00}(A) = K_0(A)$.

For all $[p] \in K_{00}$ it holds that $\pi(p) = 0$. Thus, the induced map $\psi\colon K_{00}(A^+) \to K_0(A)$ acts as follows:

$$\psi([p] - [q]) = [p] + [p_0] - [q] + [p_0] = [p] - [q] \ ,$$

$$\Rightarrow \quad \psi = \mathrm{Id} \quad \Rightarrow \quad \omega_A = \mathrm{Id}_{K_{00}(A) \to K_0(A)} \ .$$

□

Theorem 3.4.53.
Let J be a closed ideal of A, $\iota\colon J \to A$ be the inclusion, and $\rho\colon A \to {}^{A}\!/\!{}_{J}$ be the canonical projection. Then the sequence

$$K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\rho_*} K_0({}^{A}\!/\!{}_{J})$$

is exact, i.e. $Im(\iota_*) = \mathrm{Ker}(\rho_*)$.

Proof 3.4.54.
Let $[p] - [p_n] \in K_0(J)$, where $p - p_n \in M_k(J)$. Then:

$$\rho_*(\iota_*([p] - [p_n])) = [\rho(p)] - [\rho(p_n)] = [\rho(p_n)] - [\rho(p_n)] = 0 \ .$$

In the second step, we used that for $\iota(a)$, where $a \in J$, it holds that $\rho(\iota(a)) = [\iota(a)] = [0]$ in ${}^{N}\!/\!{}_{J}$. This shows that $\mathrm{Im}(\iota_*) \subset \mathrm{Ker}(\rho_*)$.

On the other hand, let $[p] - [p_n] \in \mathrm{Ker}(\rho_*)$; again, such that $p - p_n \in M_k(J)$. By theorem 3.4.40 (iii), there is an $r \geq k + m$ and a $u \in U_r(({}^{A}\!/\!{}_{J})^+) \subset M_r(({}^{A}\!/\!{}_{J})^+)$, such that

$$u \ \mathrm{diag}(\rho(p), p_n, 0)u^* = \mathrm{diag}(p_n, p_m, 0) \quad \text{in } M_{2r}(({}^{A}\!/\!{}_{J})^+) \ .$$

[4]Caution/Reminder: A^+ is the set $A \times \mathbb{C}$, yet the multiplicative structure is not component wise as in $A \oplus \mathbb{C}$.

From corollary 3.3.35 it follows that there is a unitary homotopy between $\mathrm{diag}(\mathbb{1}, \mathbb{1}) = \mathbb{1}$ and $\mathrm{diag}(u, u^*)$. Hence $\mathrm{diag}(u, u^*) \in U_{2r}((A/J)^+)_0$, where the zero means the connected component of the unit element. Because of corollary 3.2.26, there is $v \in U_{2r}(A^+)$, such that $\rho(v) = \mathrm{diag}(u, u^*)$. Define

$$f := v\ \mathrm{diag}(p, p_m, 0)v^* \quad \in M_{2r}(A^+)\ ,$$

then it holds that $\rho(f) = p_{n+m}$, i.e. $f \in M_{2r}(J^+)$ and $f - p_{n+m} \in M_{2r}(J)$. Summarizing all results, in $K_0(A^+)$ it holds that

$$[p] - [p_n] = [\mathrm{diag}(p, p_m, 0)] - [p_{n+m}] = [f] - [p_{n+m}] \in \mathrm{Im}(\iota_*)\ .$$

Thus $\mathrm{Ker}(\rho_*) \subset \mathrm{Im}(\iota_*)$. □

One might think, that adding zeros at the end of the sequences of theorem 3.4.53 could make the sequence a short exact sequence. However, we have not shown that $\mathrm{Ker}(\iota_*) = 0$ and $\mathrm{Im}(\rho_*) = K_0(A/J)$. In fact this does not always hold true, such that K_0 is not exact (cf. [Bla86, p. 5.6.2]).

3.5 Higher K groups

Higher K-groups will be defined inductively from a relation between K_1 and K_0. To define K_0, some results about $\mathrm{GL}_n(A)$ and $U_n(A)$ are needed first, especially the definitions, in the non-unital case.

3.5.1 GL_n and U_n

If R is a unital algebra, then $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with coefficients in R. If R has no unit, $\mathrm{GL}_n(R)$ is not well defines. Since A is not unital in general, we make the following definition:

Definition 3.5.1.
We define for a non-unital A:

$$\mathrm{GL}'_n(A) := \{u \in \mathrm{GL}_n(A^+) \mid u \equiv \mathbb{1}_n \mod M_n(A)\} ,$$

$$U'_n(A) := U_n(A^+) \cap \mathrm{GL}'_n(A) .$$

A priori, $\mathrm{GL}_n(A)$ and $\mathrm{GL}'_n(A)$ are not the same. However, if A is unital, they become isomorphic.

Lemma 3.5.2.
Let A be unital, then $\mathrm{GL}_n(A) \cong \mathrm{GL}'_n(A)$ as topological groups.

Proof 3.5.3.
First we show, that $M_n(A^+) \cong M_n(A)^+$. Let $x = (x_{ij})$ and $y = (y_{ij}) \in \mathrm{GL}_n(A^+)$. Then, because of the coefficient wise definition, there are $a = (a_{ij}), b = (b_{ij}) \in M_n(A)$, such that

$$x_{ij} = (a_{ij}, z_{ij}) \quad \text{and} \quad y_{ij} = (b_{ij}, w_{ij}) ,$$

where $z_{ij}, w_{ij} \in \mathbb{C}$. Consider the map[5]

$$\varphi\colon M_n(A^+) \longrightarrow M_n(A)^+ , \quad (a_{ij}, z_{ij}) \longmapsto (a_{ij}) + (z_{ij}) .$$

Injectivity and surjectivity follow immediately. It remains to show that φ is a group morphism (linearity is immediate from the coefficient wise definition):

$$\begin{aligned}(\varphi(x)\varphi(y))_{ij} &= \sum_k \varphi(x)_{ik}\varphi(y)_{kj} = \sum_k (a_{ik} + z_{ik})(b_{kj} + w_{kj}) \\ &= \sum_k a_{ik}b_{kj} + z_{ik}b_{kj} + a_{ik}w_{kj} + z_{ik}w_{kj} \\ &= (an + zb + aw)_{ij} + (zw)_{ij} \\ &= \varphi(xy) .\end{aligned}$$

This shows that φ is indeed an isomorphism.

Let $u \in \mathrm{GL}'_n(A)$, then it can be written as $\varphi(u) = a + \mathbb{1}_n$. Furthermore, there is an $v \in \mathrm{GL}_n(A^+)$, such that $uv = \mathbb{1}_n$ with $\varphi(v) = b + z$.

$$\mathbb{1}_n = uv = ab + b + az + z \;,$$
$$\Rightarrow \quad z = \mathbb{1}_n \quad \text{and} \quad ab + b + a = 0 \;.$$

Let $\mathtt{1}_n$ denote the matrix with $\mathtt{1} \in A$ on the diagonal.

$$ab + b + a = 0 \quad \Leftrightarrow \quad \mathtt{1}_n = ab + b + a + \mathtt{1}_n = (a + \mathtt{1}_n)(b + \mathtt{1}_n)$$
$$\Leftrightarrow \quad (a + \mathtt{1}_n) \in \mathrm{GL}_n(A) \;.$$

Hence, we can define the following map:

$$\psi \colon \mathrm{GL}'_n(A) \longrightarrow \mathrm{GL}_n(A) \;, \quad u = a + \mathbb{1}_n \longmapsto a + \mathtt{1}_n \;.$$

Note, that we have implicitly used the isomorphism φ for the construction of ψ.

Injectivity of ψ: Let $u = a + \mathbb{1}_n, v = b + \mathbb{1}_n \in \mathrm{GL}'_n(A)$ and assume that $\psi(u) = \psi(v)$. Then it holds that $a + \mathtt{1}_n = b + \mathtt{1}_n$, thus $a = b$ and hence $u = v$.

Surjectivity of ψ: Let $x \in \mathrm{GL}_n(A)$ and define $a \coloneqq x - \mathtt{1}_n$. Since $x \in \mathrm{GL}_n(A)$, there is a $y \in \mathrm{GL}_n(A)$, such that $xy = yx = \mathtt{1}_n$. Define $b \coloneqq y - \mathtt{1}_n$, then we calculate:

$$ab + a + b = 0 = ba + a + b$$
$$\Rightarrow \quad (a + \mathbb{1}_n)(b + \mathbb{1}_n) = \mathbb{1}_n = (b + \mathbb{1}_n)(a + \mathbb{1}_n) \;.$$

This shows that $a + \mathbb{1}_n \in \mathrm{GL}'_n(A)$ and

$$\psi(a + \mathbb{1}_n) = a + \mathtt{1}_n = x - \mathtt{1}_n + \mathtt{1}_n = x \;.$$

morphism property of ψ:

$$\psi(a + \mathbb{1}_n)\psi(b + \mathbb{1}_n) = (a + \mathtt{1}_n)(b + \mathtt{1}_n) = ab + a + b + \mathtt{1}_n$$
$$= \psi(ab + a + b + \mathbb{1}_n) = \psi((a + \mathbb{1}_n)(b + \mathbb{1}_n)) \;.$$

Continuity of ψ: We observe, that $u - v = a + \mathbb{1}_n - b - \mathbb{1}_n = a - b$ and

$$\psi(u - v) = \psi(u) - \psi(v) = a + \mathtt{1}_n - b - \mathtt{1}_n = a - b \;.$$
$$\Rightarrow \quad \|u - v\| = \|a - b\| = \|\psi(u) - \psi(v)\| \;.$$

This is enough to show continuity for both ψ and ψ^{-1}. □

Corollary 3.5.4.
It holds that $M_n(A^+) = M_n(A)^+$.

Proof 3.5.5.
This has also been shown in the proof of lemma 3.5.2.

[5]Here we use the notation, indicated in remark 3.4.33, i.e. $((a_{ij}), (z_{ij}) \equiv (a_{ij}) + (z_{ij})$. We do so, as the wrong interpretation $(a_{ij}) + (z_{ij})\mathtt{1}_{M_n(A)}$ is unnatural here anyway.

Corollary 3.5.6.
If A is unital, it holds that $U'_n(A) \cong U_n(A)$.

Proof 3.5.7.
Consider the isomorphism

$$\psi\colon \mathrm{GL}'_n(A) \longrightarrow \mathrm{GL}_n(A)\ , \quad u = a + \mathbb{1}_n \longmapsto a + \mathbf{1}_n\ .$$

Assume that $u = a + \mathbb{1}_n \in U'_n(A)$, i.e. $u \in U_n(A^+)$. Then it holds has to hold that

$$\mathbb{1}_n \stackrel{!}{=} (a + \mathbb{1}_n)(a + \mathbb{1}_n)^* = (a + \mathbb{1}_n)(a^* + \mathbb{1}_n) = aa^* + a + a^* + \mathbb{1}_n\ ,$$

$$\mathbb{1}_n \stackrel{!}{=} (a + \mathbb{1}_n)^*(a + \mathbb{1}_n) = (a + \mathbb{1}_n)(a^* + \mathbb{1}_n) = a^*a + a + a^* + \mathbb{1}_n\ ,$$

$$\Leftrightarrow \quad aa^+a + a^* = 0 = a^*a + a + a^*\ .$$

$$\Rightarrow \quad \psi(u)\psi(u)^* = \mathbf{1}_n = \psi(u)^*\psi(u)\ .$$

The opposite direction is similar. Hence, the restriction of ψ is an isomorphism for $U'_n(A) \cong U_n(A)$. □

Because of the isomorphisms in the unital case, we use $\mathrm{GL}_n(A)$ and $U_n(A)$ for $\mathrm{GL}'_n(A)$ and $U'_n(A)$. As before, let $\mathrm{GL}_\infty(A)_0$ denote the connected component of the unit element.

Remark 3.5.8.
As can be seen in [Weg93, lemma 4.2.1 and propositino 4.2.4] (and some results of topology), the connected components coincide with the path connected components for $\mathrm{GL}_n(A)$ and $U_n(A)$.

Lemma 3.5.9.
It holds that

$$\mathrm{GL}_n(A) \cap U_n(A) = U_0(A)\ .$$

Proof 3.5.10.
"$\subset$" the direction $U_0 \subset \mathrm{GL}_n(A)_0 \cap U_n(A)$ is immediate.
"$\supset$" Consider the map $r\colon g \mapsto g|g|^{-1}$. From the proof of theorem 3.3.27 we know that indeed $r\colon \mathrm{GL}_n(A) \to U_n(A)$. Next, we want to show that r is continuous. Since product, inversion (because of the von Neumann series) and the $*$-map are already continuous, it remains to show that $\sqrt{\cdot}\colon \mathrm{GL}_n(A)_+ \to \mathrm{GL}_n(A)_+$ is continuous (where $+$ denotes the set of positive elements from definition 2.4.1).

Let $x \geq 0$, then for all $y \geq 0$ with $\|y\| \leq \|x\| + 1$ it holds that $\sigma(y) \subset [0, \|x\| + 1]$. Hence, by lemma 2.3.20 $\sqrt{\cdot}$ is continuous in x. Since we made no restrictions for x, the map $\sqrt{\cdot}$ is continuous in all points $x \in \mathrm{GL}_n(A)_+$ and thus continuous on $\mathrm{GL}_n(A)_+$ by lemma 1.2.1.

So $r\colon g \mapsto g|g|^{-1}$ is continuous, and furthermore $r(u) = u$ for $u \in U_n(A)$. This means that $r|_{U_n(A)} = \mathrm{Id}_{U_n(A)}$ and $r(\mathrm{GL}_n(A)_0) = U_n(A)_0$. But then, for all $u \in \mathrm{GL}_n(A)_0 \cap U_n(A)$ it holds that $u = r(u) \subset U_n(A)_0$, so $\mathrm{GL}_n(A) \cap U_n(A) \subset U_n(A)_0$. $\square$

Lemma 3.5.11.
For $g \in \mathrm{GL}_n(A)$ it holds that $|g|, |g|^{-1} \in \mathrm{GL}_n(A)_0$.

Proof 3.5.12 (similar to [Weg93, proof of lemma 4.2.3])**.**
It holds that $|g|^{-1} \in \mathrm{GL}_n(A)$ with $|g|$ as inverse (follows form using the functional calculus for $C^*(|g|, \mathbb{1})$, since $\mathrm{GL}_n(A)$ is already unital). Since $|g| > 0$ we can define the path

$$p_t := g|g|^{-1}\exp(t\ln(|z|)) \quad \text{with inverse} \quad p_t^{-1} = \exp(-t\ln(|z|))(g|g|^{-1})^* \ .$$

So p_t is a continuous path in $\mathrm{GL}_n(A)$ from $p_0 = g|g|^{-1}$ to $p_1 = g$. Then, because of the continuity of products, and since $g \in \mathrm{GL}_n(A)$, the path $g^{-1}p_t$ is a continuous path in $\mathrm{GL}_n(A)$ from $g^{-1}p_0 = |g|^{-1}$ to $g^{-1}p_1 = \mathbb{1}$. Hence $|g|^{-1} \in \mathrm{GL}_n(A)_0$. Since $\mathrm{GL}_n(A)_0$ is a proper sub group and with the uniqueness of the inverse and unit, it follows that $(|g|^{-1})^{-1} = |g| \in \mathrm{GL}_n(A)_0$. $\square$

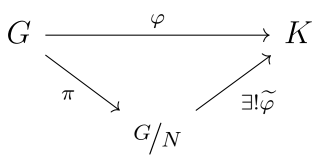
Remark 3.5.13 (Reminder)**.**
From algebra, we recall, that a normal subgroup $N \triangleleft G$ of a group G is a subgroup $N \subset G$, such that left cosets equal the corresponding right cosets $gN = Ng$. The following properties are equivalent:

$$N \triangleleft G \quad \Leftarrow \quad gNg^{-1} \subset N \ , \ \forall\, g \in G \quad \Leftrightarrow \quad gNg^{-1} = N \ , \ \forall\, g \in G \ .$$

The **quotient group** is

$$G/N := \{gN \mid g \in G\} \quad \text{with} \quad gN \circ hN = (gh)N \ .$$

The neutral element is $N = eN = nN$ for all $n \in N$ and the inverse element of gN is $g^{-1}N$.
The quotient group has the following universal property (known as **fundamental homomorphism theorem**): Let $\varphi\colon G \to K$ be a group morphism and $N \subset \mathrm{Ker}(\varphi)$, then there exists a unique group morphism $\widetilde{\varphi}\colon G/N \to K$, such that the following diagram commutes:



For $K = \mathrm{Im}(\varphi)$ and $N = \mathrm{Ker}(\varphi)$ it holds that $\widetilde{\varphi}\colon \mathrm{Ker}(\varphi) \to \mathrm{Im}(\varphi)$ is an isomorphism.

Lemma 3.5.14.
For all $n \in \mathbb{N}$ and $n = \infty$, it holds that $\mathrm{GL}_n(A)_0 \lhd \mathrm{GL}_n(A)$ *and* $U_n(A)_0 \lhd U_n(A)$.

Proof 3.5.15.
Let $g \in \mathrm{GL}_n(A)_0$, and consider the conjugation map $c_g\colon h \longmapsto ghg^{-1}$. Since $\mathrm{GL}_n(A)$ is a topological group,$c_g\colon \mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_n(A)$ is continuous. Hence, c_g maps connected components to connected components. From $c_g(\mathbb{1}) = \mathbb{1}$ it thus follows that $c_g(\mathrm{GL}_n(A)_0) \subseteq \mathrm{GL}_n(A)_0$.

In the same way, $c_g(U_n(A)) \subset U_n(A)_0$ for $g \in U_n(A)$. □

Lemma 3.5.16.
For all $n \in \mathbb{N}$ the following holds:

$$\mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0 \cong U_n(A)/U_n(A)_0 \ .$$

Proof 3.5.17.
Let $\iota\colon U_n(A) \hookrightarrow \mathrm{GL}_n(A)$ be the inclusion and $\pi\colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0$ as well as $\pi'\colon U_n(A) \to U_n(A)/U_n(A)_0$ be the projections to the quotient group. Since $\iota(U_n(A)_0) \subset \mathrm{GL}_n(A)_0$, it holds that $U_n(A)_0 \subseteq \mathrm{Ker}(\pi \circ \iota)$. Then, by the fundamental homomorphism theorem, there is a unique group morphism $\varphi\colon U_n(A)/U_n(A)_0 \to \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0$, such that the following diagram commutes:

$$\begin{array}{ccc} U_n(A) & \overset{\iota}{\hookrightarrow} & \mathrm{GL}_n(A) \\ {\scriptstyle \pi'}\downarrow & \overset{\pi\circ\iota}{\searrow} & \downarrow{\scriptstyle \pi} \\ U_n(A)/U_n(A)_0 & \underset{\exists!\ \varphi}{\longrightarrow} & \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0 \end{array}$$

It remains to show that φ is bijective. To differentiate the coset classes we write $[\cdot]_U$ and $[\cdot]_G$.
Injectivity: Let $[u]_U$, such that $\varphi([u]_U) = [\mathbb{1}]_G$. Then:

$$[\mathbb{1}]_G = \varphi([u]_U) = \varphi(\pi'(u)) = (\varphi \circ \pi')(u) = (\pi \circ \iota)(u)$$

This means that $u \in \mathrm{GL}_n(A)_0 \cap U_n(A)$, so $u \in U_n(A)_0$ by lemma 3.5.9. So $[u]_U = [\mathbb{1}]_U$, which shows that φ is injective.
Surjectivity: Let $[g]_G \in \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0$ and $u := g|g|^{-1} \in U_n(A)$. By lemma 3.5.11 it follows that $|g|^{-1} \in \mathrm{GL}_n(A)_0$. So $\pi(|g|^{-1}) = [\mathbb{1}]_G$. Since $g|g|^{-1} \in U_n(A)$ (proof of theorem 3.3.27) we define $u = g|g|^{-1} \in U_n(a)$ and obtain:

$$\begin{aligned} [g]_G = [g]_G[\mathbb{1}]_G &= \pi(g)\pi(|g|^{-1}) = \pi(g|g|^{-1}) = \pi(u) = (\pi \circ \iota)(u) \\ &= \varphi(\pi'(u)) = \varphi([u]_U) \ . \end{aligned}$$

This shows that φ is surjective. □

Definition 3.5.18.
We denote the direct limit of the system $(\mathrm{GL}_n(A), \phi_{mn})$, where

$$\phi_{n,n+1} \colon M_n(A) \longrightarrow M_{n+1}(A) \ , \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

with $\mathrm{GL}_\infty(A) := \varinjlim \mathrm{GL}_n(A)$. In the same way, we define $U_\infty(A)$.
We equip $\mathrm{GL}_\infty(A)$ and $U_\infty(A)$ with the **limit topology**, i.e. $U \subset \mathrm{GL}_\infty(A)$ is open, if $U \cap \mathrm{GL}_n(A)$ is open for all $n \in \mathbb{N}$.

For the next lemma, we need a result from topology:

Remark 3.5.19.
Let G be a topological group and $H \subset G$ be an open subgroup. Then gH for $g \not\in H$ is also open. The complement $H^c = \bigcup_{g \notin H} gH$ is open, so H is closed. Hence H is closed in G and thus the union of connected components of G. On the other hand, a result of topology/group theory is, that the connected component of the unit element G_0 is a sub group of G. Hence, it is the smallest open sub group.

Lemma 3.5.20.
It holds that $(\mathrm{GL}_n(A)_0, \phi_{mn})$ *and* $(U_n(A)_0, \phi_{mn})$ *are direct systems with direct limits* $(\mathrm{GL}_\infty(A)_0, \Phi_n)$ *and* $(U_\infty(A)_0, \Phi_n)$.

Proof 3.5.21.
Œ we consider only $\mathrm{GL}_n(A)_0$ here, since Φ_n and ϕ_{mn} commute with the $*$-map.
The maps ϕ_{mn} are continuous and unital, such that

$$\phi_{mn} \colon \mathrm{GL}_m(A)_0 \longrightarrow \mathrm{GL}_n(A)_0 \ .$$

Hence $(\mathrm{GL}_n(A)_0, \phi_{mn})$ is a well defined direct system. Since the ϕ_{mn} are injective, it follows from corollary 3.1.15, that the Φ_n are injective. They are also continuous and as group morphisms unital, such that

$$\Phi_n \colon \mathrm{GL}_n(A)_0 \longrightarrow \mathrm{GL}_\infty(A)_0 \ .$$

Let now $G = \varinjlim \mathrm{GL}_n(A)_0$ be the direct limit of $(\mathrm{GL}_n(A)_0, \phi_{mn})$ with maps $\Psi_n \colon \mathrm{GL}_n(A)_0 \to G$, then by definition of the direct limit, there is a unique morphism $\xi \colon G \to \mathrm{GL}_\infty(A)_0$, such the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{GL}_m(A)_0 & \xrightarrow{\phi_{mn}} & \mathrm{GL}_n(A)_0 & \xrightarrow{\Psi_n} & G \\ & {\scriptstyle \Phi_m} \searrow & \downarrow {\scriptstyle \Phi_n} & \swarrow {\scriptstyle \exists!\ \xi} & \\ & & \mathrm{GL}_\infty(A)_0 & & \end{array}$$

It remains to show that ξ is bijective.

By definition of the topology of $\mathrm{GL}_\infty(A)$, the direct limit G (constructed as in theorem 3.1.10) is an open subgroup of $\mathrm{GL}_\infty(A)_0$. However, since $\mathrm{GL}_\infty(A)$ is a topological group, $\mathrm{GL}_\infty(A)_0$ is the smallest sub group of $\mathrm{GL}_\infty(A)$, so $G = \mathrm{GL}_\infty(A)_0$ by remark 3.5.19. Hence, the inclusion $\xi \equiv \iota \colon G \longmapsto \mathrm{GL}_\infty(A)_0$ is a surjective group morphism. Furthermore, as inclusion, it is injective, such that ξ is a group isomorphism.

In the same way, one sees that $U_\infty(A)_0 = \varinjlim U_n(A)_0$. □

Corollary 3.5.22.

i) *Let* $u_0, u_1 \in \mathrm{GL}_n(A)$, *such that* $\Phi_n(u_0 u_1^{-1}) \in \mathrm{GL}_\infty(A)_0$, *then there is an* $\mathbb{N} \ni m \geq n$, *such that* $\phi_{nm}(u_0 u_1^{-1}) \in \mathrm{GL}_m(A)_0$

ii) *Let* $u_0, u_1 \in U_n(A)$, *such that* $\Phi_n(u_0 u_1^*) \in U_\infty(A)_0$, *then there is an* $\mathbb{N} \ni m \geq n$, *such that* $\phi_{nm}(u_0 u_1^*) \in U_m(A)_0$

The same holds for $u_1^{-1} u_0$ *in both cases.*

Proof 3.5.23.
Since $u_1^* = u_1^{-1}$ we only need to consider the unitary case. From lemma 3.5.20 together with corollary 3.1.12 it follows that there is an $k \in \mathbb{N}$ and $x_k \in U_k(A)_0$, such that $\Phi_k(x_k) = \Phi_n(u_0 u_1^*)$. Thus:

$$\Phi_m(\phi_{km}(x_k)) = \Phi_m(\phi_{nm}(u_0 u_1^*))$$

Since Φ_m is injective and $\phi_{km}(x_k) \in U_m(A)_0$ it holds that

$$\phi_{nm}(u_0 u_1^*) = \phi_{km}(x_k) \in U_m(A)_0 \ .$$

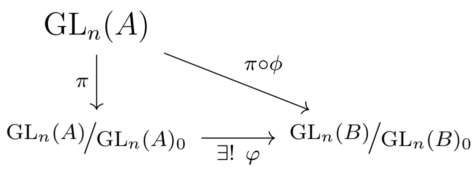
□

Lemma 3.5.24.
Let $\phi \colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(B)$ *be a continuous group morphism for any* $n \in \mathbb{N}$ *or* $n = \infty$. *Then there is a unique group morphism* $\varphi \colon {}^{\mathrm{GL}_n(A)}/_{\mathrm{GL}_n(A)_0} \to {}^{\mathrm{GL}_n(B)}/_{\mathrm{GL}_n(B)_0}$, *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{GL}_n(A) & \xrightarrow{\ \phi\ } & \mathrm{GL}_n(B) \\ \pi \downarrow & & \downarrow \pi \\ {}^{\mathrm{GL}_n(A)}/_{\mathrm{GL}_n(A)_0} & \xrightarrow{\ \exists!\ \varphi\ } & {}^{\mathrm{GL}_n(B)}/_{\mathrm{GL}_n(B)_0} \end{array}$$

The same also holds for U_n.

Proof 3.5.25.
Since ϕ is continuous, it holds that $\phi(\mathrm{GL}_n(A)_0) \subseteq \mathrm{GL}_n(B)_0$. Thus $\mathrm{GL}_n(A)_0 \subset \mathrm{Ker}(\pi \circ \phi)$. The rest follows from the fundamental homomorphism theorem

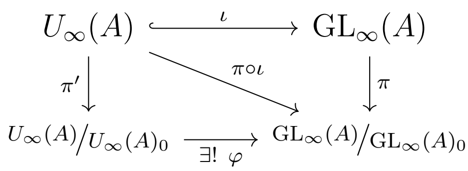


The proof for U_n is exactly the same. □

Lemma 3.5.26.
Similar to lemma 3.5.16 it holds that

$$\mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0 \cong U_\infty(A)/U_\infty(A)_0 \ .$$

Proof 3.5.27.
As before, the following diagram commutes, and there is a unique morphism by the fundamental homomorphism theorem:



Again, it remains to show that φ is bijective.
Injectivity: Let $u \in U_\infty(A)$ and assume that $\varphi([u]_U) = [\mathbb{1}]_G$. This implies as before, that $u \in \mathrm{GL}_\infty(A)_0$. There are $k, m \in \mathbb{N}$ as well as $u'_m \in U_m(A)$ and $u_k \in \mathrm{GL}_k(A)_0$, such that

$$\Phi_m(u'_m) = u = \Phi_k(u_k) \ .$$

But then, because of the injectivity of Φ_n it holds that

$$\Phi_n(\phi_{mn}(u'_m)) = \Phi_n(\phi_{kn}(u_k)) \quad \Rightarrow \quad v := \phi_{mn}(u'_m) = \phi_{kn}(u_k) \ .$$

Since $\phi_{mn}\colon \mathrm{GL}_m(A)_0 \to \mathrm{GL}_n(A)_0$ but also $\phi_{kn}\colon U_k(A) \to U_n(A)$. So $v \in \mathrm{GL}_n(A)_0 \cap U_n(A) = U_n(A)_0$ by lemma 3.5.9. Hence, since $\Phi_n\colon U_n(A)_0 \to U_\infty(A)_0$, it holds that $u = \Phi_n(\phi_{kn}(u_k)) = \Phi_n(v) \in U_\infty(A)_0$. So $[u]_U = [\mathbb{1}]_U$, which shows injectivity.
Surjectivity: Let $g \in \mathrm{GL}_\infty(A)$, then there is an $n \in \mathbb{N}$, and a $g_n \in \mathrm{GL}_n(A)$, such that $\Phi_n(g_n) = g$. Let $u_n = g_n|g_n|^{-1} \in U_n(A)$, then by lemma 3.5.11 it we see that

$$u_n^{-1} g_n = |g_n| g_n^{-1} g_n = |g_n| \in \mathrm{GL}_n(A)_0 \ .$$

Thus $u := \Phi_n(u_n) \in U_\infty(A)$, $\Phi_n(u_n^{-1} g_n) \in \mathrm{GL}_\infty(A)_0$ i.e. $[\Phi_n(u_n^{-1} g_n)]_G = [\mathbb{1}]_G$ and $\Phi_n(u_n)\Phi_n(u^{-1} g) = \Phi_n(g)$. Hence:

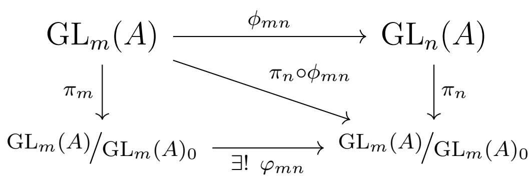
$$\varphi([u]_U) = \pi(u) = \pi(u)\pi(u^{-1}g) = \pi(g) = [g]_G \ .$$

□

Theorem 3.5.28.
It holds that

$$\lim_{\rightarrow} {}^{\mathrm{GL}_n(A)}\!/\!_{\mathrm{GL}_n(A)_0} \cong \lim_{\rightarrow} {}^{U_n(A)}\!/\!_{U_n(A)_0}$$
$$\cong {}^{\mathrm{GL}_\infty(A)}\!/\!_{\mathrm{GL}_\infty(A)_0} \cong {}^{U_\infty(A)}\!/\!_{U_\infty(A)_0} \; .$$

Proof 3.5.29.
Since $\phi_{mn}\colon \mathrm{GL}_m(A)_0 \to \mathrm{GL}_n(A)_0$ it holds that $\phi_{mn}(\mathrm{GL}_m(A)_0) \subset \mathrm{GL}_n(A)_0$. With $\pi_n(\mathrm{GL}_n(A)_0) \subset [\mathbb{1}]_n$, so $\mathrm{GL}_m(A)_0 \subset \mathrm{Ker}(\pi_n \circ \phi_{mn})$. By the fundamental homomorphism theorem there is a unique group morphism $\varphi_{mn}\colon {}^{\mathrm{GL}_m(A)}\!/\!_{\mathrm{GL}_m(A)_0} \to {}^{\mathrm{GL}_n(A)}\!/\!_{\mathrm{GL}_n(A)_0}$, such that the following diagram commutes:



So it hols that $\varphi_{mn} \circ \pi_m = \phi_{mn} \circ \pi_n$. With

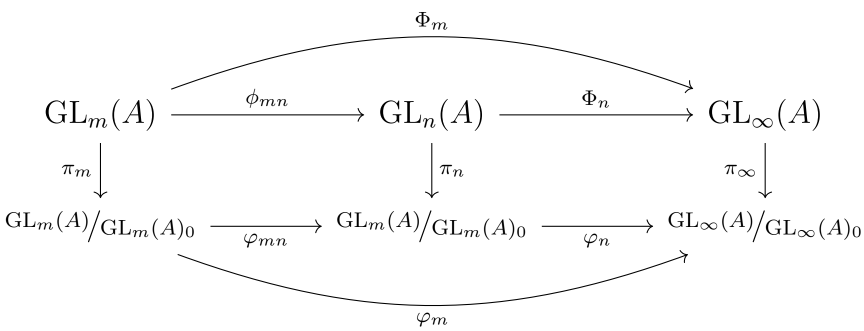
$$\begin{aligned}\varphi_{nk} \circ \varphi_{mn} \circ \pi_m &= \varphi_{nk} \circ \pi_n \circ \phi_{mn} = \pi_k \circ \phi_{nk} \circ \phi_{mn} = \pi_k \circ \phi_{mk} \\ &= \varphi_{mk} \circ \pi_m\end{aligned}$$

it follows that $({}^{\mathrm{GL}_n(A)}\!/\!_{\mathrm{GL}_n(A)_0}, \varphi_{mn})$ is a well defined direct system.

Since $\Phi_n(\mathrm{GL}_n(A)_0) \subset \mathrm{GL}_\infty(A)_0$ it follows in the same way from the fundamental homomorphism theorem, that there is a unique group morphism $\varphi_n\colon {}^{\mathrm{GL}_n(A)}\!/\!_{\mathrm{GL}_n(A)_0} \to {}^{\mathrm{GL}_\infty(A)}\!/\!_{\mathrm{GL}_\infty(A)_0}$ with $\varphi_n \circ \pi_n = \pi_\infty \circ \Phi_n$. Furthermore, it holds that

$$\begin{aligned}\varphi_n \circ \varphi_{mn} \circ \pi_m &= \varphi_n \circ \pi_n \circ \phi_{mn} = \pi_\infty \circ \Phi_n \circ \phi_{mn} = \pi_\infty \circ \Phi_m \\ &= \varphi_m \circ \pi_m \;,\end{aligned}$$

so all in all, the following diagram commutes:



This means, that $({}^{\mathrm{GL}_\infty(A)}\!/\!_{\mathrm{GL}_\infty(A)_0}, \varphi_n)$ satisfies the mapping property of the direct limit. Let now $(\bigsqcup_n {}^{gl_n(A)}\!/\!_{\mathrm{GL}_n(A)_0}, \Psi_n)$ be the direct limit of $({}^{\mathrm{GL}_n(A)}\!/\!_{\mathrm{GL}_n(A)_0}, \varphi_{mn})$, as constructed in theorem 3.1.10. Then there is a unique morphism
$\xi\colon G \to {}^{\mathrm{GL}_\infty(A)}\!/\!_{\mathrm{GL}_\infty(A)_0}$, such that

$$\begin{array}{ccc} \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0 & \xrightarrow{\Psi_n} & G \\ & \searrow^{\varphi_n} \quad \swarrow_{\exists!\,\xi} & \\ & \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0 & \end{array}$$

The map Ψ_n is given by $\pi_n(g_n) = [g_n]_n \mapsto [[g_n], n]$ and ξ is given by

$$[[g_n], n] \mapsto \varphi_n([g_n]) = \varphi_n(\pi(n)(g_n)) = \pi_\infty(\Phi_n(g_n)) = [\Phi_n(g_n)]_\infty \ .$$

To see that ξ is well defined, let $[h_m]_m \in [[g_n]_n, n]$, i.e. $[h_m]_m \sim [g_n]_n$ which by definition means:

$$\exists\ k \geq m, n\colon \quad [\phi_{mk}(h_m)]_k = \varphi_{mk}([h_m]_m) \stackrel{!}{=} \varphi_{nk}([g_n]_n) = [\phi_{nk}(g_n)]_k \ .$$

We have to show that $\varphi_m([h_m]_m) = \varphi_n([g_n]_n)$. However, this holds true, because

$$\begin{aligned} \varphi_m([h_m]_m) &= (\varphi_k \circ \varphi_{mk})([h_m]_m) = \varphi_k([\phi_{mk}(h_m)]_k) = \varphi_k([\phi_{nk}(g_n)]_k) \\ &= (\varphi_k \circ \varphi_{nk})([g_n]_n) = \varphi_n([g_n]_n) \ . \end{aligned}$$

So ξ is well defined and by construction

$$(\xi \circ \Psi_n)([g_n]_n) = \xi([[g_n]_n, n]) = \varphi_n([g_n]_n) \quad \Rightarrow \quad \xi \circ \Psi_n = \varphi_n \ .$$

As usual, it remains to show that ξ is bijective.
Surjectivty: Let $[g]_\infty \in \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0$. Then there are an $n \in \mathbb{N}$ and $[g_n]_n \in \mathrm{GL}_n(A)/\mathrm{GL}_n(A)_0$, such that $\varphi_n([g_n]_n) = [g]_\infty$. It follows that

$$\xi(\Psi_n([g_n]_n)) = \xi([[g_n]_n, n]) = \varphi_n([g_n]_n) = [g] \ ,$$

which shows surjectivity.
Injectivity: Assume that $\xi(\Psi_n([g_n]_n)) = \xi(\Psi_n([h_n]_n))$, then it holds that

$$\varphi_n([g_n]_n) = [\Phi_n(g_n)]_\infty = [\Phi_n(h_n)]_\infty = \varphi_n([h_n]_n) \ .$$

By definition of the cosets this means:

$$\begin{aligned} &\Phi_n(g_n)\mathrm{GL}_\infty(A)_0 = \Phi_n(h_n)\mathrm{GL}_\infty(A)_0 \\ \Rightarrow \quad &\exists\ x \in \mathrm{GL}_\infty(A)_0\colon\ \Phi_n(g_n)x = \Phi_n(h_n) \ . \end{aligned}$$

Then by lemma 3.5.20 there is an $m \in \mathbb{N}$ and $x_m \in \mathrm{GL}_m(A)_0$, such that $\Phi_m(x_m) = x$. So

$$\Phi_n(g_n)\Phi_m(x_m) = \Phi_n(h_n) \ .$$

Let $k \geq m, n$, then

$$\Phi_k(\phi_{nk}(g_n)\phi_{mk}(x_m)) = \Phi_k(\phi_{nk}(h_n)) \ .$$

Since Φ_k is injective:

$$\phi_{nk}(g_n) \underbrace{\phi_{mk}(x_m)}_{:= x_k \in \mathrm{GL}_k(A)_0} = \phi_{nk}(h_n) \ .$$

With $[x_k]_k = [\mathbb{1}]_k$ it follows that

$$\begin{aligned} &\varphi_{nk}([g_n]_n) = [\phi_{nk}(g_n)]_k = [\phi_{nk}(g_n)][x_k] = [\phi_{nk}(h_n)] = \varphi_{nk}([h_n]_n) \\ \Rightarrow \quad &[g_n]_n \sim [h_n]_n \quad \Rightarrow \quad \Psi_n([g_n]) = [[g_n]_n, n] = [[h_n]_n, n] = \Psi_n([h_n]_n) \ . \end{aligned}$$

This shows injectivity.

The proof for $U_n(A)$ is similar. The isomorphy follows from lemma 3.5.26. $\square$

3.5.2 The K_1-group

Definition 3.5.30.
The ***K_1*-group** is defined as quotient group:

$$K_1(A) := {}^{\mathrm{GL}_\infty(A)}/_{\mathrm{GL}_\infty(A)_0} \ .$$

Because of theorem 3.5.28, we could have used any of these quotients to define K_1. Depending on the claim to prove, any of these quotients will be used, whichever sees fit.

Remark 3.5.31.
In the following sections, elements of different K-groups will appear at the same time. For that reason we write $[\cdot]_k$ for elements of the k-th K-group.

Lemma 3.5.32.
The group $K_1(A)$ is commutative, where the product is defined by

$$[u]_1[v]_1 = [uv]_1 := [\mathrm{diag}(u,v)]_1 \ .$$

Proof 3.5.33 (from [Weg93, proof of proposition 7.1.2])**.**
The definition of the multiplication is well defined, because of remark 3.5.8 and the following equivalence

$$u \sim_h u' \text{ and } v \sim_h v' \quad \Leftrightarrow \quad uv \sim_h u'v' \ .$$

Hence, if $u' \in [u]_1$ and $v' \in [v]_1$, it holds that $u'v' \in [uv]_1$.
From corollary 3.3.35 it follows that

$$[uv]_1 = [\mathrm{diag}(uv, \mathbb{1})]_1 = [\mathrm{diag}(u,v)]_1 = [\mathrm{diag}(v,u)]_1 = [vu]_1 \ .$$

□

Lemma 3.5.34 (**Functoriality of K_1**)**.**
A $*$-morphism $\xi\colon A \to B$ induces a morphism $\xi_*\colon K_1(A) \to K_1(B)$.

Proof 3.5.35.
By theorem 2.6.10, ξ is continuous. Define $\xi_n\colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(B)$ components wise, then the ξ_n are group morphisms and again continuous.

Consider now the direct systems $(\mathrm{GL}_n(A), \phi_{mn})$ and $(\mathrm{GL}_n(B), \psi_{mn})$ defined as in definition 3.5.18. The inclusions ϕ_{mn} do not depend on the algebra, so in fact $\phi_{mn} = \psi_{mn}$. Since $\mathrm{GL}_n(A) \subset \mathrm{GL}_n(A^+)$ and $\xi((0,1)) = (\xi(0),1) = (0,1)$ it follows that $\phi_{mn} \circ \xi_n = \xi_m \circ \phi_{mn}$. By lemma 3.1.19 there is a unique morphism $\zeta\colon \mathrm{GL}_\infty(A) \to \mathrm{GL}_\infty(B)$. Since GL_∞ is quipped with the limit topology, the map ζ is continuous (by its construction $\zeta(\Phi_n(a_n)) = \Phi_n(\xi_n(a_n))$ in lemma 3.1.19).

From lemma 3.5.24 it follows, that there is a unique map

$$\xi_*\colon {}^{\mathrm{GL}_\infty(A)}/{}_{\mathrm{GL}_\infty(A)_0} = K_1(A) \longrightarrow m^{\mathrm{GL}_\infty(B)}/{}_{\mathrm{GL}_\infty(B)_0} = K_1(B)\ ,$$

such that $\pi \circ \xi_* = \zeta \circ \pi$. □

Remark 3.5.36.
The induced morphism from lemma 3.5.34 is constructed as follows. Let $[g]_1 \in {}^{\mathrm{GL}_\infty(A)}/{}_{\mathrm{GL}_\infty(A)_0} = K_1(A)$ with $g_n \in \mathrm{GL}_n(A)$ such that $g = \Phi_n(g_n)$. Then if $\xi_n\colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(B)$ denotes the component wise extension of $\xi\colon A \to B$ it holds that:

$$\xi_*([g]_1) = [\xi(g)]_1 = [\Phi_n(\xi_n(g_n))]_1 \in {}^{\mathrm{GL}_\infty(B)}/{}_{\mathrm{GL}_\infty(B)_0} = K_1(B)\ .$$

3.5.3 Suspension and higher K-groups

Definition 3.5.37.
The **suspension** SA of a local C^*-algebra A is defined as

$$SA := C_0((0,1), A) \cong \{f \in C(\mathbb{S}^1, A) \mid f(1) = 0\}\ .$$

The operations are defined point wise, and SA is equipped with the sup-norm.

With the operations defined point wise, and the sup-norm, SA is a local C^*-algebra, if A is a local C^*-algebra. $\mathbb{S}^1 \subset \mathbb{C}$ is the unit circle in $\mathbb{C}$. Since $(0,1)$ and $\mathbb{S}^1 \setminus 1$ are homeomorphic, such that

$$C_0((0,1), A) \cong C_0(\mathbb{S}^1 \setminus 1, A)\ .$$

Then $\mathbb{S}^1$ is obtained from $\mathbb{S}^1 \setminus 1$ by a one-point compactification. It can be shown, that the functions of $C_0(\mathbb{S}^1 \setminus 1, A)$ are those functions, that are in $C_0(\mathbb{S}^1, A)$ with $f(1) = 0$.

Corollary 3.5.38 (Functoriality of S).
Every $*$-morphism $\phi\colon A \longrightarrow B$ induces a $*$-morphism $S\phi\colon SA \longrightarrow SB$ by

$$S\phi\colon f \longmapsto \phi \circ f\ .$$

Proof 3.5.39.
Let $f, g \in SA$, and $\diamond$ a shorthand notation for the algebra operation then, for all $b \in B$ it holds that

$$\begin{aligned} S\phi(f \diamond g)(b) &= \phi((f \diamond g)(b)) = \phi(f(b) \diamond g(b)) = \phi(f(b)) \diamond \phi(g(b)) \\ &= S\phi(f)(b) \diamond S\phi(g)(b) = (S\phi(f) \diamond S\phi(g))(b)\ . \end{aligned}$$

For the $*$-map, it follows that

$$S\phi(f)(b)^* = (\phi(f(b)))^* = \phi(f(b)^*) \equiv \phi(\overline{f}(b)) = S\phi(\overline{f})(b)\ .$$

□

Remark 3.5.40.
A result from analysis is, that continuity is a component wise property, such that

$$SM_n(A) \cong M_n(SA) \ .$$

The functions correspond to each other by $f((a_{ij})) \leftrightarrow (f_{ij}(a_{ij}))$.

Remark 3.5.41.
The elements of $(SA)^+$ are $f \oplus z$ with $(f,z)(t) = (f(t), z) =: f(t) + z$ for $f \in C(\mathbb{S}^1, A)$ with $f(1) = 0$ and $z \in \mathbb{C}$. Hence $g := (f,z)$ is a function in $C(\mathbb{S}^1, A^+)$ with $g(1) = z$ and for all $t \in \mathbb{S}^1$ it holds that $g(t) = f(t) + z \equiv x_t + z$ with $x_t \in A$. Hence:

$$\begin{aligned}(SA)^+ &\cong \{f \in C(\mathbb{S}^1, A^+) \mid f(1) = \lambda \in \mathbb{C} \ , \ \forall\, t\, \exists\, x_t \in A \colon f(t) = \lambda + x_t\} \\ &\cong \{f \in C([0,1], A^+) \mid f(0) = f(1) = \lambda \in \mathbb{C} \ , \ \forall\, t \exists\, x_t \in A \colon f(t) = \lambda + x_t\} \ .\end{aligned}$$

This means, that elements of (SA^+) are loops in $f \in A^+$, that are constantly $f(1)$ modulo A. Furthermore, it follows that

$$S(A^+) \subset (SA)^+ \ .$$

Remark 3.5.42.
Let $u \in U_k(A)$, then by adding $1 \in \mathbb{C}$, on the diagonal, using the maps $\phi_{k,k+1}$ as defined in definition 3.5.18, it is possible to increase k, whenever necessary. Formally this means, that if $\Phi_k(u) = u_\infty \in U_\infty(A)$, it holds that $\Phi_{k+m}(\mathrm{diag}(u, \mathbb{1}_m)) = u_\infty$. Furthermore, because of corollary 3.3.35, we find

$$\mathrm{diag}(u, \mathbb{1}_{nk}) \sim_h \mathrm{diag}(\mathbb{1}_{nk}, u) \quad \forall\, n \in \mathbb{N} \ .$$

Lemma 3.5.43.
Let $v \in U_n(A)$ and $w \in U_{k-n}(A)$ for $k > n \in \mathbb{N}$. Then, there is a $k' \geq k \in \mathbb{N}$, such that

$$\mathrm{diag}(v, w, \mathbb{1}_{k'-k}) \sim_h \mathrm{diag}(w, \mathbb{1}_{k'-k}, v) \ .$$

and

$$\mathrm{diag}(v, \mathbb{1}_{k'-n}) \sim_h \mathrm{diag}(\mathbb{1}_{k'-n}, v) \ .$$

Proof 3.5.44.
There are three cases. First if $k-n < n$, there is an $m \in \mathbb{N}$, such that $k+m-n = n$, since $k, n \in \mathbb{N}$. It remains to define $k' = k + m$ such that $k' - n = n$. Then

$\operatorname{diag}(w, \mathbb{1}_{k'-k}) \in U_{k-n+k'-k}(A) = U_n(A)$. The rest follows from corollary 3.3.35:

$$\begin{array}{ccc} \operatorname{diag}(v, \operatorname{diag}(w, \mathbb{1}_{k'-k})) & \sim_h & \operatorname{diag}(\operatorname{diag}(w, \mathbb{1}_{k'-k}), v) \\ \| & & \| \\ \operatorname{diag}(v, w, \mathbb{1}_{k'-k}) & & \operatorname{diag}(w, \mathbb{1}_{k'-k}, v) \end{array} . \tag{3.3}$$

The case $k - n = n$ follows immediately.

In the last case $n < k-n$. So let $a \in \mathbb{N}$, such that $k-n = n+a$. Let $i = n\cdot(k-n)$ and

$$j = i - a = n \cdot (k - n) - a = n \cdot (n + a) - a \in \mathbb{N} .$$

From remark 3.5.42 it follows that

$$\operatorname{diag}(w, \mathbb{1}_i) \sim_h \operatorname{diag}(\mathbb{1}_i, w) \quad \text{and} \quad \operatorname{diag}(v, \mathbb{1}_i) \sim_h \operatorname{diag}(\mathbb{1}_i, v) . \tag{3.4}$$

Furthermore, by construction

$$n + i = n + j + a = n + a + j = k - n + j$$

so in the same way as in(3.3)

$$\operatorname{diag}(v, \mathbb{1}_i, w, \mathbb{1}_j) \sim_h \operatorname{diag}(w, \mathbb{1}_j, v, \mathbb{1}_i) . \tag{3.5}$$

So choosing $k' = k + i + j$ and combining (3.4) and (3.5) we obtain the claim:

$$\begin{aligned} \operatorname{diag}(v, w, \mathbb{1}_{k'-k}) =& \operatorname{diag}(v, w, \mathbb{1}_{i+j}) = \operatorname{diag}(v, w, \mathbb{1}_i, \mathbb{1}_j) \\ &\sim_h \operatorname{diag}(v, \mathbb{1}_i, w, \mathbb{1}_j) \sim_h \operatorname{diag}(w, \mathbb{1}_j, v, \mathbb{1}_i) \\ &\sim_h \operatorname{diag}(w, \mathbb{1}_j, \mathbb{1}_i, v) \sim_h \operatorname{diag}(w, \mathbb{1}_{i+j}, v) \\ &= \operatorname{diag}(w, \mathbb{1}_{k'-k}, v) . \end{aligned}$$

□

Theorem 3.5.45.
For all local C^*-algebras A there is an isomorphism $\theta_A \colon K_1(A) \to K_0(SA)$, such that for all $*$-morphisms $\phi \colon A \to B$ the following diragram commutes:

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\phi_*} & K_1(B) \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ K_0(SA) & \xrightarrow[(S\phi)_*]{} & K_0(SB) \end{array}$$

The proof of this theorem is rather involved. To separate the main steps, from the subtle details and auxiliary remarks, the latter are printer in gray. However, this does not make the proof any shorter, unfortunately.

Proof 3.5.46.
Construction of $\boldsymbol{\theta_A}$: Let $u \in U_n(A)$ and v_t be a continuous path from $\mathbb{1}$ to $\operatorname{diag}(u, u^*)$ in $U_{2n}(A)$, as in corollary 3.3.35. Note, that actually $u \in U'_n(A)$

so by definition $u \in U_n(A^+)$ with $u \equiv \mathbb{1}_n \mod M_n(A)$ etc. . Recall that $p_n = \text{diag}(1, \dots, 1, 0, \dots, 0) \in U_{2n}(A)$ and define

$$p_t := v_t p_n v_t^* \quad \Rightarrow \quad p_0 = p_1 = p_n \quad \text{and} \quad p_t \equiv p_n \mod M_{2n}(A) \ .$$

Furthermore it holds that $p_t^2 = p_t^* = p_t$ so $p_t \in \text{Proj}(M_{2n}(A^+))$. Considering the path as set of component functions $p\colon t \mapsto p_t$ we see that (since $p_0 = p_1 = p_n \in M_{2n}(\mathbb{C})$)

$$p \in \text{Proj}(M_{2n}((SA)^+)) \ .$$

Hence[6], $[p]_{00} \in K_{00}((SA)^+)$ and understanding p_n as constant loop $p_n\colon t \mapsto p_n$ it also holds that $[p_n]_{00} \in K_{00}((SA)^+)$. By theorem 3.4.40 (ii) it holds that $[p]_{00} - [p_n]_{00} \in K_0(SA)$, such that we can define:

$$\theta_A([u]_1) := [p]_{00} - [p_n]_{00} \in K_0(SA) \ .$$

Well definedness of $\boldsymbol{\theta_A}$: Let $[u']_1 = [u]_1$. Because of corollary 3.5.22, there is a sufficiently large $n \in \mathbb{N}$, such that there are paths a_t and b_t in $U_n(A)$ from $\mathbb{1}_n$ to $(u')^* u$ and from $\mathbb{1}_n$ to $u' u^*$. Recall, that v_t is a continuous path from $\mathbb{1}$ to $\text{diag}(u, u^*)$, and define v_t' to be a continuous path from $\mathbb{1}$ to $\text{diag}(u', (u')^*)$. So

$$p_t = v_t p_n v_t^* \quad \text{and define} \quad p_t' = v_t' p_n (v_t')^* \ .$$

Also, define $x_t := v_t' \, \text{diag}(a_t, b_t) v_t^*$, then

$$x_0 = \mathbb{1} \ , \quad x_1 = \text{diag}(u', (u')^*) \text{diag}((u')^* u, u' u^*) \text{diag}(u^*, u) = \mathbb{1} \ .$$

Furthermore, $x_t \equiv \mathbb{1} \mod M_{2n}(A)$ (this can be seen by corollary 3.5.4, which allows to write all terms in the product as $M + \mathbb{1}$ leading to something of the form $M' + \mathbb{1}$). Again, understanding $x\colon t \mapsto x_t$ as set of component functions, it holds that

$$x \in U_{2n}((SA)^+) \ .$$

For the functions $x\colon t \mapsto x_t$ and $p\colon t \mapsto p_t$ we calculate (using that $v_t \in U_n(A)$, $\text{diag}(a_t, b_t) \in U_{2n}(A)$) and for all $M = \text{diag}(X, Y) \in M_{2n}(A^+)$ it holds that $M p_n = p_n M$ since $p_n = (\mathbb{1}, 0) \in M_{2n}(A^+)$):

$$(x p x^*)_t = v_t' \, \text{diag}(a_t, b_t) \; p_n \; \text{diag}(a_t^*, b_t^*) (v_t')^* = v_t' p_n (v_t')^* = p_t' \ .$$

Hence $p_t \sim_u p_t'$, that is $[p_t]_{00} = [p_t']_{00}$ for all t, so $[p]_{00} = [p']_{00}$, where again $p'\colon t \mapsto p_t'$ and thus

$$[p]_{00} - [p_n]_{00} = [p']_{00} - [p_n]_{00} \ .$$

This shows, that θ_A is well defined.

$\boldsymbol{\theta_A}$ is a group morphism: Let $[u]_1, [v]_1 \in K_1(A)$, Œ with $u, v \in U_n(A)$ and $x_t, y_t \in U_{2n}A$ be continuous paths from $\mathbb{1}$ to $\text{diag}(u, u^*)$ and $\text{diag}(v, v^*)$. Define again $p\colon t \mapsto p_t = a_t p_n a_t^*$ and $q\colon t \mapsto q_t = b_t p_n b_t^*$. Then by construction, $\theta_A([u]_1) = [p]_{00} - [p_n]_{00}$ and $\theta_A([v]_1) = [q]_{00} - [p_n]_{00}$. With the addition in $V(A)$ from lemma 3.4.2 it follows that, $[p]_{00} + [q]_{00} = [\text{diag}(p, q)]_{00}$ for $M_{2n}(A)$ (or larger

[6] Again, sloppy notation! More carefully, one would have to write $\Phi_{2n}(p)$ etc.

dimension) and also $[p_n]_{00} + [p_n]_{00} = [p_{2n}]_{00}$. Note that $\mathrm{diag}(p_t, q_t)$ is a path from $\mathbb{1}$ to $\mathrm{diag}(\mathrm{diag}(u, u^*), \mathrm{diag}(v, v^*)) = \mathrm{diag}(u, u^*, v, v^*)$. Since $[u, v]_1 = \mathrm{diag}(u, v)$ by lemma 3.5.32 we obtain:

$$\begin{aligned}\theta_A([u]_1[v]_1) &= \theta_A([\mathrm{diag}(u, v)]_1) = [\mathrm{diag}(p, q)]_{00} - [p_{2n}]_{00}\\ &= ([p]_{00} + [q]_{00}) - ([p_n]_{00} + [p_n]_{00})\\ &= ([p]_{00} - [p_n]_{00}) + ([q]_{00} - [p_n]_{00})\\ &= \theta_A([u]_1) + \theta_A([v]_1)\ .\end{aligned}$$

The diagram commutes: Let $[u]_1 \in K_1(A)$ with $u \in U_n(A)$, then (in a sloppy notation) $\phi_*([u]_1) = [\phi(u)]_1$ by remark 3.5.36. On the other hand

$$(S\phi)_*([p]_{00} - [p_n]_{00}) = (S\phi)_*([p]_{00}) - (S\phi)_*([p_n]_{00}) = [\phi \circ p]_{00} - [\phi \circ p_n]_{00}\ ,$$

by remark 3.4.22 and corollary 3.5.38. Furthermore, p denotes the map $t \mapsto p_t$, so $\phi \circ p$ denotes $t \mapsto \phi(p_t)$, which we can denote as $\phi(p) = \phi \circ p$. All in all, we see that:

$$\begin{aligned}\theta_B(\phi_*([u]_1)) &= \theta_B([\phi(u)]_1) = [\phi(p)]_{00} - [\phi(p_n)]_{00} = [\phi \circ p]_{00} - [\phi \circ p_n]_{00}\\ &= (S\phi)_*([p]_{00} - [p_n]_{00}) = (S\phi)_*(\theta_A([u]_1))\ .\end{aligned}$$

Injectivity: Let $\theta_A([u]_1) = 0$, where u and v_t, p_t are defined as in the first part of the proof. From theorem 3.4.40 (iii) it follows (Recall, that $[p]_{00} - [p_n]_{00} \in K_0(SA)$), that for sufficiently large $k \in \mathbb{N}$ there is a $w \in U_k((SA)^+)$, such that

$$w\ \mathrm{diag}(p, p_m)w^* = \mathrm{diag}(p_n, p_m)$$

Furthermore, $p\colon t \mapsto p_t = v_t p_n v_t^*$. Let now $v\colon t \mapsto v_t$ denote the path, then $p = vpv^*$ and thus

$$\begin{aligned}w\ \mathrm{diag}(p, p_m)w^* &= w\ \mathrm{diag}(v, \mathbb{1})\mathrm{diag}(p_n, p_m)\mathrm{diag}(v^*, \mathbb{1})w^*\ .\\ &= \mathrm{diag}(p_n, p_m)\end{aligned}$$

This means that $w\ \mathrm{diag}(v, \mathbb{1})$ has the form $\mathrm{diag}(a, b)$ for paths $a\colon t \mapsto a_t$ and $b\colon t \mapsto b_t$, where $a \in U_n((SA)^+)$ and $b \in U_{k-n}((SA)^+)$ This can be seen as follows. A priori $w\ \mathrm{diag}(v, \mathbb{1})$ is a block matrix of the form $\mathrm{diag}(\alpha, \beta)$ of the form with $\alpha \in M_{2n}$ and $\beta \in M_{k-2n}$ because $\mathrm{diag}(v, \mathbb{1})$ is of this form. However in order for α and p_n, which is $\mathbb{1}$ on the upper $n \times n$-block, to commute, α has to be of the form $\alpha = \mathrm{diag}(a, \gamma)$ and similarly $\beta = \mathrm{diag}(\delta, \epsilon)$. With $b = \mathrm{diag}(\gamma, \delta, \epsilon)$, it follows that

$$w\ \mathrm{diag}(v, \mathbb{1}) = \mathrm{diag}(a, b)\ .$$

Since w is a path $w\colon t \mapsto w_t$ in $U_k((SA)^+)$, it holds that

$$w_0 = w_1 = z \in M_k(\mathbb{C}) \quad \Rightarrow \quad w_t \equiv z \mod M_k(A)\ .$$

The path $w \in U_k((SA)^+)$ has been introduced, such that

$$w^*\ \mathrm{diag}(p, p_m)w^* = \mathrm{diag}(p_n, p_m)$$

$$\Rightarrow \quad 1w_t \operatorname{diag}(p_t, p_m) w_t = \operatorname{diag}(p_n, p_m) \ .$$

Because of $p_0 = p_1 = p_n$, it follows in the same way as before, that

$$w_0 = w_1 = z = \operatorname{diag}(z', z'') \quad \text{with } z' \in U_n(\mathbb{C}),\ z'' \in U_{n-k}(\mathbb{C}) \ .$$

Then, since $v_0 p_n v_0^* = \mathbb{1}$ and $v_1 p_n v_1^* = \operatorname{diag}(u, u^*) \in U_{2n}(A)$, we obtain

$$a_0 = z' \quad \text{and} \quad a_1 = z' u \ ,$$

However, multiplying from left with z'^{-1}, $z'^{-1} a_t$ is a continuous path in $U_n(A)$ from $\mathbb{1}$ to u. So $u \in U_n(A)_0$ and thus $[u]_1 = [\mathbb{1}]$.

Surjectivity: Let $[\Phi_k(p)]_{00} - [\Phi_k(p_n)]_{00} \in K_0(SA)$ with

$$p\colon t \mapsto p_t \in \operatorname{Proj}(M_k((SA)^+)) \quad \text{and} \quad p \equiv p_n \mod M_k(SA) \ .$$

The choice of k has no upper bound. In fact, because of $\Phi_{k'} \circ \phi_{kk'} = \Phi_k$ we can increase k arbitrarily. This freedom of choice will be used later on, to simplify the notation.

Then $p_0 = p_1 = p_n$ and $p_t \equiv p_n \mod M_k(A)$. Recall that $p\colon t \mapsto p_t$ is a homotopy between $p_0 = p_1 = p_n$. From corollary 3.3.31 it follows that there is a continuous path $u_t \in U_k(A)$ with $u_0 = \mathbb{1}$ and $u_t p_n u_t^* = p_t$. For $t = 1$ this reads

$$u_1 p_n u_1^* = p_1 = p_n \quad \Rightarrow \quad u_1 p_n = p_n u_1 \ .$$

Hence u_1 has the form $\operatorname{diag}(v, w)$ with $v \in U_n(A)$ and $w \in U_{k-n}(A)$.

Next we increase k to k', and define $\widetilde{w} = \operatorname{diag}(w, \mathbb{1}_{k'-k})$. It follows that $\widetilde{w}^* = \widetilde{w^*}$. Because of lemma 3.5.43, k' can be chosen, such that

$$\operatorname{diag}(v^*, \widetilde{w}^*) \sim_h \operatorname{diag}(\widetilde{w}^*, v^*) \ .$$

Yet $\operatorname{diag}(u_t, \mathbb{1}_{k'-k})$ is a continuous path form

$$u_0 = \operatorname{diag}(\mathbb{1}_k, \mathbb{1}_{k'-k}) = \mathbb{1}_{k'} \quad \text{to} \quad \operatorname{diag}(v, w, \mathbb{1}_{k'-k}) = \operatorname{diag}(v, \widetilde{w}) \ .$$

So, together with the continuity of the $*$-map:

$$\operatorname{diag}(v, \widetilde{w}) \sim_h \mathbb{1}_{k'} \quad \Rightarrow \quad \operatorname{diag}(\widetilde{w}^*, v) \sim_h \operatorname{diag}(v^*, \widetilde{w}^*) \sim_h \mathbb{1}_{k'}^* = \mathbb{1}_{k'} \ .$$

Multiplying with $\operatorname{diag}(\widetilde{w}, \mathbb{1}_n)$ from the left, we get

$$\operatorname{diag}(\mathbb{1}_{k'-n}, v^*) \sim_h \operatorname{diag}(\widetilde{w}, \mathbb{1}_n) \ .$$

Adding $k' - 2n$ ones on the diagonal, leads to

$$\operatorname{diag}(\mathbb{1}_{k'-n}, v, \mathbb{1}_{k'-2n}) \sim_h \operatorname{diag}(v, \mathbb{1}_{2k'-3n}) \sim_h \operatorname{diag}(\widetilde{w}, \mathbb{1}_{k'-n})$$

Defining $k'' = 2k' - n$ and $\widehat{w} := \operatorname{diag}(w, \mathbb{1}_{k'-n})$ we obtain

$$\operatorname{diag}(v^*, \mathbb{1}_{k''-2n}) \sim_h \widehat{w} \ ,$$

where $\widehat{w} \in U_{k''-n}(A)$. Since we could have chosen k to be k'' in the first place by $\Phi_{k''} \circ \phi_{k,k''} = \Phi_k$, we simply write

$$\operatorname{diag}(v^*, \mathbb{1}_{k-2n}) \sim_h w \quad \text{in} \quad U_{k-n}(A) \ .$$

Let b_t be a continuous path in $U_{k-n}(A)$ from w to $\operatorname{diag}(v^*, \mathbb{1}_{k-2n})$ and again by corollary 3.3.35 z_t be a continuous path from $\mathbb{1}$ $\operatorname{diag}(v, v^*, \mathbb{1}_{k-2n})$. Define $q_t \colon z_t p_n z_t^*$, then this is a continuous path in $U_k(A)$ from p_n to $\operatorname{diag}(v, 0)$, such that

$$\theta_A([v]_1) = [q]_{00} - [p_n]_{00} \ ,$$

where $q \colon t \mapsto q_t$. Now define $x_t := u_t \operatorname{diag}(\mathbb{1}_n, w^* b_t) z_t^*$, then $x_0 = \mathbb{1}_k$ and

$$x_1 = \operatorname{diag}(v, w) \operatorname{diag}(\mathbb{1}_n, w^* \operatorname{diag}(v^*, \mathbb{1}_{k-2n})) \operatorname{diag}(v^*, v, \mathbb{1}_{k-2n}) = \mathbb{1}_k \ .$$

This means that $x \in U_K((SA)^+)$. Considering xqx^* as path $xqx^* \colon t \mapsto (xqx^*)_t$ we find

$$\begin{aligned}
(xqx^*)_t &= u_t \operatorname{diag}(\mathbb{1}_n, w^* b_t) z_t^* z_t p_n z_t^* z_t \operatorname{diag}(\mathbb{1}, b_t^* w) u_t^* \\
&= u_t \operatorname{diag}(\mathbb{1}_n, w^* b_t) p_n \operatorname{diag}(\mathbb{1}, b_t^* w) u_t^* \\
&= u_t p_n \operatorname{diag}(\mathbb{1}_n, w^* b_t) \operatorname{diag}(\mathbb{1}, b_t^* w) u_t^* \\
&= u_t p_n u_t^* = p_t \ .
\end{aligned}$$

So $xqx^* = p$, with $x \in U_K((SA)^+)$, which means $q \sim_u p$, which by definition means $[q]_{00} = [p]_{00}$. Hence we have $\theta_A([v]_1) = [q]_{00} - [p_n]_{00} = [p]_{00} - [p_n]_{00}$, which shows surjectivity. □

Corollary 3.5.47.
Let $u_t \in U_k(A)$ be a continuous path from $u_0 = \mathbb{1}_k$ to $u_1 = \operatorname{diag}(v, w)$, where $v \in U_n(A)$ and $w \in U_{k-n}(A)$. Then by increasing k to k' and identifying w with $\operatorname{diag}(w, \mathbb{1}_{k'-n})$ it holds that

$$\operatorname{diag}(v^*, \mathbb{1}_{k'-2n}) \sim_h w \ .$$

Proof 3.5.48.
This has been shown in the surjectivity part of the proof of theorem 3.5.45. □

Lemma 3.5.49.
The functor S is additive and commutes with direct limits over $\mathbb{N}$.

Proof 3.5.50.

i) First we observe, that continuity in $A_1 \oplus A_2 = A_1 \times A_2$ is a component wise property. So

$$f \colon \mathbb{S}^1 \longrightarrow A_1 \oplus A_2 \ , \quad t \longmapsto f(t) = (f_1(t), f_2(t)$$

is continuous, if $f_i\colon \mathbb{S}^1 \to A_i$ are continuous and wise versa. So

$$(f_1, f_2) \in C(\mathbb{S}^1, A_1 \oplus A_2) \quad \text{and} \quad f_i := \pi_i \circ f \in C(\mathbb{S}^1, A_i) \ ,$$

where $\pi_i\colon A_1 \oplus A_2 \to A_i$ is the canonical projection. Furthermore

$$f(1) = (0,0) \quad \Leftrightarrow \quad f_i(1) = 0 \ .$$

This shows that $S(A_1 \oplus A_2) = S(A_1) \oplus S(A_2)$.

ii) Let (A, Φ_n) be the direct limit of (A_n, ϕ_{mn}), then, because of functoriality $(SA, S\Phi_n)$ satisfies the mapping property for $(SA_n, S\phi_{mn})$ (cf. proof of theorem 3.4.12). Let $(\varinjlim SA_n, \Psi_n)$ be the inductive limit, then there is a unique morphism $\varphi\colon \lim \varinjlim SA_n \to SA$, such that the following diagram commutes:

$$\begin{array}{ccc} SA_n & \xrightarrow{\Psi_n} & \varinjlim SA_n \\ {\scriptstyle S\Phi_n}\downarrow & \swarrow {\scriptstyle \exists! \varphi} & \\ SA & & \end{array}$$

As usual, it has to be shown, that φ is bijective.

Injectivity: Let $f \in \varinjlim SA_n$, such that $f = \Psi_n(f_n)$ and assume $\varphi(f) = 0 = S\Phi_n(f_n)$. This means that

$$\forall\, t \in (0,1)\colon S\Phi_n(f_n)(t) = \Phi_n(f_n(t)) = 0$$

$$\text{and} \quad S\Phi_n(f_n)(0) = S\Phi_n(f_n)(1) = 0 \ ,$$

$$\Rightarrow \quad \Phi_n(f_n(t)) = 0 \quad \forall\, t \in [0,1] \ .$$

Since ϕ_{nk} are $*$-morphisms, they are norm decreasing (theorem 2.6.10) and thus

$$\inf_{k \geq n} \|\phi_{kn}(f_n)(t)\| = \lim_{k \to \infty} \|\phi_{kn}(f_b)\| = 0 \quad \forall\, t \in [0,1] \ .$$

Define $g_k\colon \|\cdot\| \circ \phi_{nk} \circ f_n\colon [0,1] \to \mathbb{R}$, i.e. $g_k(t) = \|\phi_{nk}(f_n(t))\|$, then (g_k) is a monotonously falling series of functions $[0,1] \to \mathbb{R}$ with continuous limit function $g_\infty = 0$. By Dini's theorem (since $[0,1] \subset \mathbb{R}$ is compact) it follows that (g_k) converges uniformly against 0:

$$\Rightarrow \quad 0 = \inf_{k \geq n} \|g_m - 0\|_{\sup} = \inf_{k \geq n} \sup_{t \in [0,1]} g_k(t) \ .$$

It follows that (cf. example 3.1.6)

$$\begin{aligned} \|f\| &= \limsup_{k \geq n} \|S\phi_{nk}(f_n)\| = \inf_{k \geq n} \sup_{m \geq k} \|S\phi_{nm}(f_n)\| \\ &= \inf_{k \geq n} \|S\phi_{kn}(f_n)\| = \inf_{k \geq n} \sup_{t \in [0,1]} \|\phi_{kn}(f_n(t))\| \\ &= \inf_{k \geq n} \sup_{t \in [0,1]} g_k(t) = 0 \ , \end{aligned}$$

where we used that $\|S\phi_{nm}(f_n)\|$ is decreasing, such that the supremum can be ignored. We have obtained $\|f\| = 0$, which means that $f = 0$. So $\varphi(f) = 0$ implies $f = 0$, which implies injectivity.

Surjectivity: Let $f \in SA$, then, since $f \in C([0,1], A)$ with $f(0) = f(1) = 0$ and $[0,1]$ is compact, f is uniformly continuous. Let $\varepsilon > 0$, then f can be approximated by a polygonal chain, that is

$$\exists\, N \in \mathbb{N}\,, \quad 0 < t_0 < t_1 < \ldots < t_N < 1:$$
$$\Big\| f(t) - \underbrace{\Big(f(t_j)\tfrac{t_{j+1}-t}{t_{j+1}-t_j} + f(t_{j+1})\tfrac{t-t_j}{t_{j+1}-t_j} \Big)}_{=:\ell_j(t)} \Big\| \le \varepsilon \quad \forall\, t \in [t_j, t_{j+1}]\,.$$

There are $b_0, \ldots, b_N \in A_n$, such that $\Phi_n(b_j) = f(t_j) \in A$. Since f approaches 0 for $t \to 0, 1$, we may assume that $b_0 = b_N = 0$. Consider

$$g(t) := \begin{cases} b_j \frac{t_{j+1}-t}{t_{j+1}-t_j} + b_{j+1}\frac{t-t_j}{t_{j+1}-t_j} & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \le t_0 \text{ or } t \ge t_N \end{cases}$$

then g is continuous and thus $g \in SA_n$. It follows that

$$\begin{aligned} \varphi(\Psi_n(g))(t) &= S\Phi_n(g)(t) = \Phi_n(g(t)) \\ &= \begin{cases} \Phi_n(b_j)\frac{t_{j+1}-t}{t_{j+1}-t_j} + \Phi_n(b_{j+1})\frac{t-t_j}{t_{j+1}-t_j} & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \le t_0 \text{ or } t \ge t_N \end{cases} \\ &= \begin{cases} f(t_j)\frac{t_{j+1}-t}{t_{j+1}-t_j} + f(t_{j+1})\frac{t-t_j}{t_{j+1}-t_j} & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \le t_0 \text{ or } t \ge t_N \end{cases} \\ &= \begin{cases} \ell_j(t) & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \le t_0 \text{ or } t \ge t_N \end{cases} \end{aligned}$$

This means that

$$\begin{aligned} \|\varphi(\Psi_n(g)) - f\| = \max \Bigg\{ &\sup_{t \le t_0} \|f(t) - 0\|, \sup_{t \ge t_N} \|f(t) - 0\|\,, \\ &\max_j \Big\{ \sup_{t \in [t_j, t_{j+1}]} \|f(t) - \ell_j(t)\| \Big\} \Bigg\} \le \varepsilon \end{aligned}$$

Hence $\mathrm{Im}(\varphi) \subset SA$ densely. From corollary 2.6.12, it follows that the extension $\widehat{\varphi}$ to the completion of SA and $\varinjlim SA_n$ of φ is closed. So φ is closed for the subset topology of $SA \subset \widehat{SA}$, and thus $\mathrm{Im}(\varphi) = SA$.

□

Theorem 3.5.51.
The functor K_1 from local C^*-algebras to abelian groups is homotopy invariant, additive and commutes with direct limits over $\mathbb{N}$.

Proof 3.5.52.

i) Recall that homotopy invariance means that if $\phi_0, \phi_1 \colon A \to B$ are homotopic, then $(\phi_0)_* = (\phi_1)_*$. So let ϕ_t be a homotopy for ϕ_0 and ϕ_1. It follows that $S\phi_t$ is a homotopy for $S\phi_0$ and $S\phi_1$. By theorem 3.4.43, K_0 is homotopy invariant, such that

$$(S\phi_0)_* = (S\phi_1)_* \colon K_0(SA) \longrightarrow K_0(SB) \ .$$

From theorem 3.5.45 it follows that

$$(\phi_0)_* = \theta_B^{-1} \circ (S\phi_0)_* \circ \theta_A = \theta_B^{-1} \circ (S\phi_1)_* \circ \theta_A = (\phi_1)_* \ .$$

ii) From theorem 3.4.43 and lemma 3.5.49 we know that both K_0 and S are additive and commute with direct limits over $\mathbb{N}$. With the isomorphy of theorem 3.5.45 we find:

$$\begin{aligned} K_1(A \oplus B) &\cong K_0(S(A \oplus B)) \cong K_0(S(A) \oplus S(B)) \\ &\cong K_0(S(A)) \oplus K_0(S(B)) \cong K_1(A) \oplus K_1(B) \ . \end{aligned}$$

iii) and also:

$$\begin{aligned} K_1(\lim_{\to} A_n) &\cong K_0(S(\lim_{\to} A_n)) \cong K_0(\lim_{\to} S(A_n)) \\ &\cong \lim_{\to}(K_0(S(A_n))) \cong \lim_{\to} K_1(A_n) \ . \end{aligned}$$

□

Lemma 3.5.53.
Let $J \subset A$ be a closed ideal of a local C^*-algebra A, then $S(A/J) = S(A)/S(J)$.

Proof 3.5.54.
J has to be closed, such that A/J is a local C^*-algebra (cf. lemma 1.4.24). Let $f \in SA$ and $h \in SJ$, then $f(t) \in A$ and $h(t) \in J$ for all $t \in \mathbb{S}^1$. It follows that

$$(fh)(t) \equiv f(t)h(t) \in J \quad \text{and} \quad (hf)(t) = h(t)f(t) \in J \ ,$$

such that $fh, fh \in SJ$. So SJ is indeed an ideal of SA.

Let in the following $\pi \colon A \to A/J$ and $\Pi \colon SA \to SA/SJ$ denote the canonical projections. Consider the following map:

$$\Phi \colon SA/SJ \longrightarrow S(A/J) \ , \quad [f] \mapsto \pi \circ f \ .$$

Well definedness: The map $\pi \circ f$ is a map $\mathbb{S}^1 \to A/J$ with

$$(\pi \circ f)(1)\pi(f(1)) = \pi(0) = [0] \ .$$

Since π is continuous by definition of the quotient topology, $\pi \circ f$ is continuous (by assumption $f \in SA$). It follows that $\pi \circ f \in S({}^{A}/_{J})$. For $h \in SJ$ it follows that

$$(\pi \circ h)(t) = \pi(h(t)) = [0] \ .$$

Let now $g \in [f]$, i.e. $g \in SA$ and there is a $h \in SJ$, such that $g = f + h$, then:

$$\begin{aligned}(\pi \circ g)(t) &= \pi(g(t)) = \pi(f(t) + h(t)) = \pi(f(t)) + \pi(h(t)) \\ &= \pi(f(t)) + [0] = (\pi \circ f)(t) \ .\end{aligned}$$

So Φ is indeed well defined.
Morphism property: From the point wise definition of the operations, it follows that Φ is a $*$-morphism:

$$\begin{aligned}\Phi([f] \diamond [g])(t) &= \Phi([f \diamond g])(t) = \pi(f(t) \diamond g(t)) = \pi(f(t)) \diamond \pi(g(t)) \\ &= \big((\pi \circ f) \diamond (\pi \circ g)\big)(t)\end{aligned}$$

$$\text{and} \quad \Phi([f])^* = (\pi \circ f)^* \equiv \overline{\pi \circ f} = \pi \circ \overline{f} = \Phi(\overline{f}) \equiv \Phi(f^*) \ .$$

Inectivity: Let $\Phi([f]) = [0]$, then $\pi \circ f = [0]$. This means that $f \in SJ$, so $[f] = \Pi(f) = [0]$.
Surjectivity: Similar to the surjectivity part of proof 3.5.50, let $\widetilde{f} \in S({}^{A}/_{J})$, which is uniformly continuous for the same reason. Then, $\widetilde{f}$ can be approximated by a polygonal chain

$$\exists\ N \in \mathbb{N}\ , \quad 0 < t_0 < t_1 < \ldots < t_N < 1:$$
$$\Big\|\widetilde{f}(t) - \underbrace{\Big(\widetilde{f}(t_j)\tfrac{t_{j+1}-t}{t_{j+1}-t_j} + \widetilde{f}(t_{j+1})\tfrac{t-t_j}{t_{j+1}-t_j}\Big)}_{=:\ell_j(t)}\Big\| \leq \varepsilon \quad \forall\ t \in [t_j, t_{j+1}] \ .$$

Then, there are $b_0, \ldots, b_N \in A$, such that $\pi(b_j) = [b_j] = \widetilde{f}(t_j)$. Since $\widetilde{f}$ approaches $[0]$ for $t \to 0, 1$, we can Œ assume that $b_0, b_N = 0$. As before consider

$$g(t) \coloneqq \begin{cases} b_j\frac{t_{j+1}-t}{t_{j+1}-t_j} + b_{j+1}\frac{t-t_j}{t_{j+1}-t_j} & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \leq t_0 \text{ or } t \geq t_N \end{cases}$$

As polynomial, $g\colon [0,1] \to A\ , \quad t \mapsto g(t)$ is a continuous map. Since $g(0) = (1) = 0$, it follows that $g \in SA$. Furthermore:

$$(\pi \circ g)(t) = \pi(g(t)) = \begin{cases} \ell_j(t) & , \quad t \in [t_j, t_{j+1}] \\ 0 & , \quad t \leq t_0 \text{ or } t \geq t_N \end{cases}$$

So we have again:

$$\begin{aligned}\|\pi \circ g - \widetilde{f}\| = \max \Bigg\{ &\sup_{t \leq t_0} \|\widetilde{f}(t)\|, \sup_{t \geq t_N} \|\widetilde{f}(t)\| \ , \\ &\max_j \Bigg\{ \sup_{t \in [t_j, t_{j+1}]} \|\widetilde{f}(t) - \ell_j(t)\| \Bigg\}\Bigg\} \leq \varepsilon\end{aligned}$$

which means that $\text{Im}(\Phi) \subset S({}^{A}/_{J})$ densely. As in proof 3.5.50, Φ is closed and thus $\text{Im}(\Phi) = S({}^{A}/_{J})$. $\square$

Lemma 3.5.55.
The sequence

$$0 \longrightarrow S(J) \xrightarrow{S\iota} S(A) \xrightarrow{S\rho} S(A/J) \longrightarrow 0$$

is a short exact sequence.

Proof 3.5.56.
From the proof of lemma 3.5.53 it follows that $S(J)$ is an ideal of $S(A)$. Let $h \in SJ$, then

$$S\iota(h)(t) = \iota(h(t)) = h(t) \in A \quad \forall\, t \in \mathbb{S}^1 ,$$

so $S\iota \colon SJ \to SA$ is the inclusion.

Let $\widetilde{f} \in S(A/J)$, then by lemma 3.5.53 Œ $\widetilde{f} \in SA/SJ$ by isomorphy. Then there is an $f \in SA$, such that $\Pi(f) = \widetilde{f}$, where $\Pi \colon SA \to SA/SJ$ denotes the conical projection. Next we observe that

$$\Pi(f)(t) - \rho(f(t)) = f(t) + h(t) - (f(t) + a) = h(t) - a \in J \quad \forall\, t \in \mathbb{S}^1 ,$$

where $h \in SJ$ and $a \in J$ are arbitrary. However this means that $S\rho(f) = \rho \circ f = \Pi(f)$. Hence $S\rho \colon SA \to SA/SJ = S(A/J)$ is the canonical projection.

As inclusion $S\iota$ is injective, so $\mathrm{Ker} S\iota = 0$, and as canonical projection, $S\rho$ is surjective, so $\mathrm{Im}(S\rho) = S(A/J)$. Since $S\rho(SJ) = 0$ it also holds that $\mathrm{Im}(S\iota) = \mathrm{Ker}(S)$.

□

Theorem 3.5.57.
For every local C^*-algebra and every closed ideal $J \subseteq A$, the sequence

$$K_1(J) \xrightarrow{j_*} K_1(A) \xrightarrow{\rho_*} K_1(A/J)$$

is exact, where $j \colon J \to A$ and $\rho \colon A \to A/J$ are the inclusion and canonical projection.

Proof 3.5.58.
So with theorem 3.4.53, we find that

$$K_0(SJ) \xrightarrow{(S\iota)_*} K_0(SA) \xrightarrow{(S\rho)_*} K_0(SA/SJ) = K_0(S(S/A))$$

is exact. With theorem 3.5.45, the following diagram commutes, where $\theta_\bullet$ are isomorphisms:

$$\begin{array}{ccccc} K_1(J) & \xrightarrow{\iota_*} & K_1(A) & \xrightarrow{\rho_*} & K_1(A/J) \\ \theta_J \updownarrow & & \updownarrow \theta_A & & \updownarrow \theta_{A/J} \\ K_0(SJ) & \xrightarrow[(S\iota)_*]{} & K_0(SA) & \xrightarrow[(S\rho)_*]{} & K_0(S(A/J)) \end{array}$$

Hence, the upper row is exact, i.e. $\operatorname{Im}((S\iota)_*) = \operatorname{Ker}((S\rho)_*)$.

□

Remark 3.5.59.
Again, the functor K_1 is not exact, so adding zeros at the ends of the sequence from theorem 3.5.57, will not make it a shot exact sequence in general.

Lemma 3.5.60.
Suspension commutes with completion: $\widehat{SA} \cong S\widehat{A}$.

Proof 3.5.61.
Let $\phi\colon SA \to S\widehat{A}$ be the inclusion. It holds that ϕ is an isometry, since the inclusion $A \subset \widehat{A}$ is isometric:

$$\|f\|_{SA} = \sup_{t\in\mathbb{S}^1} \|f(t)\|_A = \sup_{t\in\mathbb{S}^1} \|f(t)\|_{\widehat{A}} = \|f\|_{S\widehat{A}} \; .$$

So ϕ is especially bounded. By the bounded linear transformation theorem, there exists a unique extension $\widehat{\phi}\colon \widehat{SA} \to S\widehat{A}$. Since also the inclusion $SA \subset \widehat{SA}$ is isometric, it follows that $\widehat{\phi}$ is an isometry. Hence:

$$\widehat{\phi}(f) = 0 \quad \Rightarrow \quad \|\widehat{\phi}(f)\| = \|f\| = 0 \quad \Rightarrow \quad f = 0 \; ,$$

which show that $\widehat{\phi}$ is injective. Similar to proofs 3.5.50 and 3.5.54, one can approximate a $g \in S\widehat{A}$ by a polygonal chain of length n, $\ell_n(t)$, such that $\operatorname{Im}(\widehat{\phi}) \subset S\widehat{A}$ densely. As before, it follows from corollary 2.6.12, that $\operatorname{Im}(\widehat{\phi}) = S\widehat{A}$, and thus $\widehat{\phi}$ is also surjective. □

Corollary 3.5.62.
It holds that $K_1(A) \cong K_1(A \otimes \mathcal{K})$.

Proof 3.5.63.
Since $S(M_n(A)) \cong M_n(SA)$ (remark 3.5.40) and since S commutes with direct limits by lemma 3.5.49 and with completions by lemma 3.5.60, it holds that

$$\begin{aligned} S(M_\infty(A)) &= S(\varinjlim M_n(A)) = \varinjlim S(M_n(A)) \cong \varinjlim M_n(SA) \\ &= M_\infty(SA) \; . \end{aligned}$$

Furthermore, S commutes with completions by lemma 3.5.60

$$\Rightarrow \quad SA \otimes \mathcal{K} = \widehat{M_\infty(SA)} \cong S(\widehat{M_\infty(A)}) \cong S(A \otimes \mathcal{K}) \; .$$

With theorem 3.5.45 and 3.4.45 we find:

$$K_1(A) \cong K_0(SA) \cong K_0(SA \otimes \mathcal{K}) \cong K_0(S(A \otimes \mathcal{K})) \cong K_1(A \otimes \mathcal{K}) \; .$$

□

With theorem 3.5.45 in mind, we make the following definition:

Definition 3.5.64.
For $n \geq 2$ we define the ***n*-th *K*-group** inductively as

$$K_n(A) \coloneqq K_{n-1}(SA) \cong K_0(S^n A) \ .$$

3.5.4 Long exact sequence

We start with the sort exact sequence

$$0 \longrightarrow J \stackrel{\iota}{\longrightarrow} A \stackrel{\rho}{\longrightarrow} {}^{A}\!/\!{}_{J} \longrightarrow 0$$

We already know, that neither K_0 nor K_1 preserve the sort exactness. However, we can construct a long exact sequence.

For the next lemma, we recall from corollary 3.2.26, that a unital surjective, bounded (i.e. continuous) morphism $\phi\colon A \to B$ maps the connected component of the unitals in A to the connected component of the unitals in B:

$$\phi(U(A)_0) = U(B)_0 \ .$$

Since the proof of corollary 3.2.26 (and the preceding lemma) only use functional calculus, the results also hold for unital local C^*-algebras. Following definition 3.5.1, we observe that $U_n(A) = U(M_n(A)_1)$, where

$$M_n^{\mathbb{1}}(A^+) \coloneqq \{u \in M_n(A^+) \mid u = \mathbb{1} \mod M_n(A)\} \ .$$

So even in the non-unital case, we have

$$\phi(U_n(A)_0) \equiv \phi^+(U(M_n^{\mathbb{1}}(A^+))_0) = U(M_n^{\mathbb{1}}(B^+))_0 \equiv U_n(B)_0 \ .$$

Lemma 3.5.65.

i) Let $u \in U_n({}^{A}\!/\!{}_{J})$, then there exists a $v \in U_{2n}(A)_0$, such that

$$\rho(v) = \operatorname{diag}(u, u^*)$$

ii) Let $u \in U_n({}^{A}\!/\!{}_{J})$. If there is a $w \in U_n(A)$, such that $u \sim_h \rho(w)$, then $u \in \rho(U_n(A))$, i.e. $\exists\ v \in U_n(A)$, such that $\rho(v) = u$.

Remark 3.5.66.
Of course $\rho\colon A \to {}^{A}\!/\!{}_{J}$ is extended to $U_{2n}(A)$ as usual, by first extending ρ to ρ^+, and then having it act component lemm:S is additive and commutes with limitswise on $M_{2n}(A^+) \to M_{2n}(({}^{A}\!/\!{}_{J})^+)$.

Proof 3.5.67.

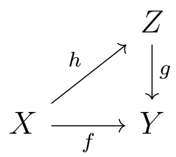
i) From corollary 3.3.35 it follows that $\mathrm{diag}(u, u^*) \sim_h \mathbb{1}$, such that $\mathrm{diag}(u, u^*) \in U_{2n}({}^{A}/_{J})_0$. It holds that $\rho(U_n(A)_0) = U_n({}^{A}/_{J})_0$, such that there is a $v \in U_{2n}(A)_0$ with $\rho(v) = \mathrm{diag}(u, u^*)$.

ii) If $u \sim_h \rho(w)$, then there is a path $p_t \in U_n({}^{A}/_{J})_0$ from u to $\rho(w)$. It follows that p_t^* is a path from u^* to $\rho(w^*)$. So up_t^* is a path from $uu^* = \mathbb{1}$ to $u\rho(w^*)$. In other words $u\rho(w^*) \in U_n({}^{A}/_{J})_0$. Since $\rho(U_n(A)_0) = U_n({}^{A}/_{J})_0$, there is a $x \in U_n(A)_0$, such that $\rho(x) = u\rho(w^*)$. This is equivalent to

$$u = \rho(x)\rho(w^*)^{-1} = \rho(x)\rho(w^{-1})^{-1} = \rho(x)\rho(w) = \rho(xw) \ .$$

Then $v = xw \in U_n(A)$. □

Remark 3.5.68.
The element v is called a **lift** of $\mathrm{diag}(u, u^*)$ and u respectively, since $\rho(v) = \mathrm{diag}(u, u^*)$ and $\rho(v) = u$ respectively. This name is inspired by the general lift. Let $f\colon X \to Y$ and $g\colon Z \to Y$ be two morphisms. A morphism $h\colon X \to Z$, that makes the diagram



is called a **lift** of f.

Definition and Lemma 3.5.69.
Let $u \in U_n({}^{A}/_{J})$ and $v \in U_{2n}(A)_0$ be a lift, i.e. $\rho(v) = \mathrm{diag}(u, u^*)$. The **index map** is defined by

$$\partial\colon K_1({}^{A}/_{J}) \longrightarrow K_0(J) \ , \quad [u]_1 \longmapsto [vp_nv^*]_{00} - [p_n]_{00} \ .$$

and is a well defined group morphism

Remark 3.5.70.
In the proof we will show, that $vp_nv^* \in J$. In J it is not generally possible to find $j \in J$, such that $jvp_nv^*j^* = p_n$, so

$$[vp_nv^*]_{00} - [p_n]_{00} \neq 0 \quad \text{in } K_0(J) \ ,$$

in general. However, considering $[vp_nv^*]_{00} - [p_n]_{00}$ in $K_0(A)$, we can choose $j = v^*$ and find

$$v^*(vp_nv^*)v = p_n \quad \Rightarrow \quad vp_nv^* \sim_u p_n$$

and thus

$$[vp_nv^*]_{00} - [p_n]_{00} = 0 \quad \text{in } K_0(A) \ .$$
$$\Rightarrow \quad \iota_* \circ \partial = 0 \ .$$

Proof 3.5.71.
Well definedness: It holds that $\rho(p_n) = p_n$, since $A^+ \ni (0,1) \overset{\rho}{\mapsto} (0,1) \in ({}^{A}/{}_{J})^+$ and thus

$$\rho(vp_nv^*) = \mathrm{diag}(u,u^*)p_n\mathrm{diag}(u^*,u) = p_n \ .$$

Hence $\rho(vp_nv^* - p_n) = p_n - p_n = 0$, which means that $vp_nv^* - p_n \in M_{2n}(J)$. However, then $vp_nv^* \in M_{2n}(J^+)$, and it follows from theorem 3.4.40 (ii) that $[vp_nv^*]_{00} - [p_n]_{00} \in K_0(J)$. So $\partial([u]_1) \in K_0(J)$ indeed.

Let now $w \in U_{2n}(A)$ be another lift of $\mathrm{diag}(u,u^*)$. Let $z := wv^{-1} \in U_{2n}(A)$, then

$$\rho(z) = \rho(wv^{-1}) = \rho(w)\rho(v)^{-1} = \mathrm{diag}(u,u^*)\mathrm{diag}(u,u^*)^{-1} = \mathbb{1}$$

and so $z \in U_{2n}(J)$. But then, since in general $[x]_{00} = [yxy^*]_{00}$ in K_{00}:

$$\begin{aligned}[vp_nv^*]_{00} - [p_n]_{00} &= [wv^{-1}vp_nv^*((v^*)^{-1}w^*)]_{00} - [p_n]_{00} \\ &= [wp_nw^*]_{00} - [p_n]_{00} = [zwp_nw^*z^*]_{00} - [p_n]_{00} \ .\end{aligned}$$

So the map ∂ does not depend on the lift.

Next let $u' \in U_n({}^{A}/{}_{J})$, such that $[u']_1 = [u]_1$ in $K_1({}^{A}/{}_{J})$. Then, $u' = uy$ for a $y \in U_n({}^{A}/{}_{J})_0$, $u * u' = u^*uy = y \in U_n({}^{A}/{}_{J})_0$ and similarly $u(u')^* \in U_n({}^{A}/{}_{J})_0$. From corollary 3.2.26 it follows that there are lifts $a, b \in U_n(A)_0$, such that $\rho(a) = u^*u'$ and $\rho(b) = u(u')^*$. We find

$$\rho(v\mathrm{diag}(a,b)) = \mathrm{diag}(u,u^*)\mathrm{diag}(u^*u', u(u')^*) = \mathrm{diag}(u', (u')^*)$$

so $v\mathrm{diag}(a,b)$ is a lift of $\mathrm{diag}(u',(u')^*)$ and because $\mathrm{diag}(a,b)$ and p_n commute:

$$[v\mathrm{diag}(a,b)p_n\mathrm{diag}(a^*,b^*)v^*]_{00} - [p_n]_{00} = [vp_nv^*]_{00} - [p_n]_{00}$$

Hence δ does not depend on the representative of $[u]$.
Morphism property: Let $u, \widetilde{u} \in U_n({}^{A}/{}_{J})$, and let $v, \widetilde{v} \in U_{2n}(A)$ be lifts, i.e. $\rho(v) = \mathrm{diag}(u,u^*)$ and $\rho(\widetilde{v}) = \mathrm{diag}(\widetilde{u},\widetilde{u}^*)$. It holds that

$$\mathrm{diag}(u,\widetilde{u},u^*,\widetilde{u}^*) = \underbrace{\begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}}_{=:w} \mathrm{diag}(u,u^*,\widetilde{u},\widetilde{u}^*) \underbrace{\begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}}_{=:w^*}$$

$$\Rightarrow \quad \rho(w \ \mathrm{diag}(v,\widetilde{v})w^*) = w \ \mathrm{diag}(u,u^*,\widetilde{u},\widetilde{u}^*)w^* = \mathrm{diag}(u,\widetilde{u},u^*,\widetilde{u}^*) \ ,$$

so $w \ \mathrm{diag}(v\widetilde{v})w^*$ is a lift of $\mathrm{diag}((u,\widetilde{u}),(u,\widetilde{u})^*) = \mathrm{diag}(u,\widetilde{u},u^*,\widetilde{u}^*)$.

Furthermore, with $w = w^*$ we also have $wp_{2n}w^* = \mathrm{diag}(p_n, p_n)$, so we find

$$\begin{aligned} &w\ \mathrm{diag}(v, \widetilde{v})w^* p_{2n} w\ \mathrm{diag}(v^*, \widetilde{v}^*)w^* \\ &\qquad = w\ \mathrm{diag}(v, \widetilde{v})wp_{2n}w^*\ \mathrm{diag}(v^*, \widetilde{v}^*)w^* \\ &\qquad = w\ \mathrm{diag}(v, \widetilde{v})\mathrm{diag}(p_n, p_n)\mathrm{diag}(v^*, \widetilde{v}^*)w^* \\ &\qquad = w\ \mathrm{diag}(vp_nv^*, \widetilde{v}p_n, \widetilde{v}^*)w^* \\ &\qquad \sim_u \mathrm{diag}(vp_nv^*, \widetilde{v}p_n, \widetilde{v}^*) \end{aligned}$$

From lemma 3.5.32 we know that $[u][\widetilde{u}] = [\mathrm{diag}(u, \widetilde{u})]$, so we can show the morphism property:

$$\begin{aligned} \partial([u]_1[\widetilde{u}]_1) &= \partial([\mathrm{diag}(u, \widetilde{u})]_1) \\ &= [w\ \mathrm{diag}(v, \widetilde{v})w^* p_{2n} w\ \mathrm{diag}(v^*, \widetilde{v}^*)w^*]_{00} - [p_{2n}]_{00} \\ &= [\sim_u \mathrm{diag}(vp_nv^*, \widetilde{v}p_n, \widetilde{v}^*)]_{00} - [\mathrm{diag}(p_n, p_n)]_{00} \\ &= [\mathrm{diag}(vp_nv^*, 0)]_{00} - [\mathrm{diag}(p_n, 0)]_{00} \\ &\qquad + [\mathrm{diag}(0, \widetilde{v}p_n\widetilde{v}^*)]_{00} - [\mathrm{diag}(0, p_n)]_{00} \\ &= [\mathrm{diag}(vp_nv^*, 0)]_{00} - [\mathrm{diag}(p_n, 0)]_{00} \\ &\qquad + [\mathrm{diag}(\widetilde{v}p_n\widetilde{v}^*, 0)]_{00} - [\mathrm{diag}(p_n, 0)]_{00} \\ &= [vp_nv^*]_{00} - [p_n]_{00} + [\widetilde{v}p_n\widetilde{v}^*]_{00} - [p_n]_{00} \\ &= \partial([u]_1) + \partial([\widetilde{u}]_1)\ . \end{aligned}$$

□

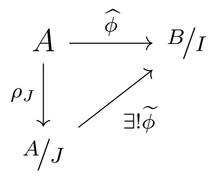
Lemma 3.5.72.
For every $*$-morphism $\phi\colon A \to B$ and all closed ideals $J \subset A$ and $I \subset B$ with $\phi(J) \subset I$, the following diagram commutes:

$$\begin{array}{ccc} K_1(A/J) & \xrightarrow{\ \widetilde{\phi}_*\ } & K_1(B/I) \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow{\scriptstyle\partial} \\ K_0(A) & \xrightarrow[\ \phi_*\ ]{} & K_0(B) \end{array}$$

Remark 3.5.73.
For $\widehat{\phi} = \rho_I \circ \phi\colon A \to B/I$ it holds that

$$\phi(J) \subset I \quad \Rightarrow \quad \widehat{\phi}(J) = \rho_I(\phi(J)) = \rho_I(I) = [0]\ ,$$

where $\rho_I\colon B \to B/I$ is the canonical projection. Then by the fundamental theorem on homomorphisms for rings, there is a unique $\widetilde{\phi}\colon A/J \to B/I$, such that the following diagram commutes:



Proof 3.5.74.
Let $u \in U_n(A/J)$ with lift $v \in U_{2n}(A)$ of $\mathrm{diag}(u, u^*)$. Then with remark 3.4.22 we find (recall from the beginning of subsection 3.4.3 that $\phi(p_n) \equiv \phi^+(p_n) = p_n$):

$$(\phi_* \circ \partial)([u]_1) = \phi_*([vp_nv^*]_{00} - [p_n]_{00}) = [\phi(v)p_n\phi(v)^*]_{00} - [p_n]_{00} \ .$$

Next we observe, that $\phi(v)$ is a lift of $\mathrm{diag}(\widetilde{\phi}(u), \widetilde{\phi}(u)^*)$:

$$\mathrm{diag}(\widetilde{\phi}(u), \widetilde{\phi}(u)^*) = \mathrm{diag}(\widetilde{\phi}(u), \widetilde{\phi}(u^*)) = \widetilde{\phi}(\mathrm{diag}(u, u^*)) \ .$$

In the remark we have seen that $\widetilde{\phi} \circ \rho_J = \widehat{\phi} = \rho_I \circ \phi$. Since $\rho(v) = \mathrm{diag}(u, u^*)$ we have:

$$\mathrm{diag}(\widetilde{\phi}(u), \widetilde{\phi}(u)^*) = \widetilde{\phi}(\mathrm{diag}(u, u^*)) = \widetilde{\phi}(\rho_J(v)) = \rho_I(\phi(v)) \ .$$

Finally we calculate with remark 3.5.36:

$$\begin{aligned}(\partial \circ \widetilde{\phi}_*)([u]_1) &= \partial(\widetilde{\phi}_*([u]_1)) = \partial([\widetilde{\phi}(u)]_1) = [\phi(v)p_n\phi(v)^*]_{00} - [p_n]_{00} \\ &= (\phi_* \circ \partial)([u]_1) \ .\end{aligned}$$

□

Theorem 3.5.75.
The following sequence is exact

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\rho_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\rho_*} K_0(A/J) \ .$$

Proof 3.5.76.
Exactness in all but $K_1(A/J)$ and $K_0(J)$ follows from theorems 3.4.53 and 3.5.57. It remains to show exactness in $K_1(A/J)$, i.e. $\mathrm{Im}(\rho_*) = \mathrm{Ker}(\partial)$ and exactness in $K_0(J)$, i.e. $\mathrm{Im}(\partial) = \mathrm{Ker}(\iota_*)$.

Exactness in $K_1(A/J)$: Let $u \in U_n(A)$, then $v := \mathrm{diag}(u, u^*)$ is a lift of $\mathrm{diag}(\rho(u), \rho(u)^*)$. Since $\mathrm{diag}(u, u^*)$ commutes with p_n, it follows that $\partial([\rho(u)]_1) = 0$, such that $\mathrm{Im}(\rho_*) \subseteq \mathrm{Ker}(\partial)$.

On the other hand, let $u \in U_n(A/J)$ with $\partial([u]_1) = 0$. Let $v \in U_{2n}(A)$ be a lift of $\mathrm{diag}(u, u^*)$, then $\partial([u]_1) = 0$ means that $[vp_nv^*]_{00} - [p_n]_{00} = 0$.

Next we define the partial isometry $w\colon vp_n \in M_{2n}(A^+)$ (cf. definition 2.9.30 and lemma 2.9.31). It holds that

$$w^*w = p_n \quad \text{and} \quad q := ww^* = vp_nv^* \ .$$

Furthermore, we observe that

$$\rho(w) = \rho(vp_n) = \mathrm{diag}(u, u^*)p_n = \mathrm{diag}(u, 0) \ .$$

Since $v \in U_{2n}(A)$ it holds that $v \equiv \mathbb{1} \mod M_{2n}(A)$ and so $q = vp_nv^* \equiv p_n \mod M_{2n}(A)$. Note that we have seen in proof 3.5.71, that $q = vp_nv^* \in M_{2n}(J^+)$. Hence $q - p_n \in M_{2n}(A)$ and from theorem 3.4.40 (iii) it follows that there is an $m \in \mathbb{N}$ and an $k \geq m + 2n$, such that

$$\mathrm{diag}(q, p_m) \sim_u \mathrm{diag}(p_n, p_m) \quad \text{in } M_k(J^+)$$

Because of theorem 3.3.27 it equivalently holds that

$$\mathrm{diag}(q, p_m) \sim_s \mathrm{diag}(p_n, p_m)$$

With theorem 3.3.15 we find that then also

$$\begin{array}{ccc} \mathbb{1} - \mathrm{diag}(q, p_m) & \sim & \mathbb{1} - \mathrm{diag}(p_n, p_m) \\ \| & & \| \\ \mathrm{diag}(\mathbb{1}_{2n} - q, \mathbb{1}_{k-2n} - p_m) & & \mathrm{diag}(\mathbb{1}_{2n} - p_n, \mathbb{1}_{k-2n} - p_m) \ . \end{array}$$

One easily checks, that these are projections, since

$$q^2 = vp_nv^*vp_nv^* = vp_n^2v^* = vp_nv^* = q$$

$$\text{and} \quad q^* = (vp_nv^*)^* = vp_nv^* = q \ .$$

However, then theorem 3.3.23 can be applied, which means that there is a partial isometry $w' \in M_k(J^+)$, such that

$$(w')^*w' = \mathrm{diag}(\mathbb{1}_{2n} - p_n, \mathbb{1}_{k-2n} - p_m)$$

$$\text{and} \quad w'(w')^* = \mathrm{diag}(\mathbb{1}_{2n} - q, \mathbb{1}_{k-2n} - p_m) \ .$$

From lemma 2.9.31 it follows that $w'(w')^*w' = w'$. With $(w')^*w'\mathrm{diag}(p_n, 0) = 0$ we also find that

$$w'\mathrm{diag}(p_n, 0) = w'(w')^*w'\mathrm{diag}(p_n, 0) = 0 \ .$$

Hence w' is of the form $w' = \mathrm{diag}(0_n, x)$. Using $\rho(q) = p_n$ (see proof 3.5.71) we obtain:

$$\begin{aligned} \mathrm{diag}(p_n, 0)\rho(w') &= \mathrm{diag}(p_n, 0)\rho(w'(w')^*w') = \mathrm{diag}(p_n, 0)\rho(w'(w')^*)\rho(w') \\ &= \mathrm{diag}(p_n, 0)\mathrm{diag}(\mathbb{1}_{2n} - p_n, \mathbb{1}_{k-2n} - p_m)\rho(w') = 0 \ . \end{aligned}$$

This means, that $\rho(w') = \mathrm{diag}(0_n, x')$ where $x' \in M_{k-n}(\mathbb{C})$ Recall that $w' \in M_k(J^+)$ and that $\rho^+(j, z) = (0, z)$ for $j \in J$.

Now define $z := \mathrm{diag}(w, p_m) + w'$. It holds that $w' \perp \mathrm{diag}(w, p_m)$, which can be seen as follows:

$$\begin{aligned} w'\mathrm{diag}(w, p_m) &= w'(w')^* w'\mathrm{diag}(vp_n, p_m) \\ &= w'\mathrm{diag}(\mathbb{1}_{2n} - p_n, \mathbb{1}_{k-2n} - p_m)\mathrm{diag}(vp_n, p_m) = 0 \end{aligned}$$

Then we can calculate:

$$\begin{aligned} z^*z &= (\mathrm{diag}(w^*, p_m) + (w')^*)(\mathrm{diag}(w, p_m) + w') \\ &= \mathrm{diag}(w^*w, p_m) + (w')^*w' \\ &= \mathrm{diag}(p_n, p_m) + \mathrm{diag}(\mathbb{1}_{2n} - p_n, \mathbb{1}_{k-2n} - p_m) \\ &= \mathbb{1}_k \end{aligned}$$

and also

$$\begin{aligned} zz^* &= (\mathrm{diag}(w, p_m) + w')(\mathrm{diag}(w^*, p_m) + (w')^*) \\ &= \mathrm{diag}(ww^*, p_m) + w'(w')^* \\ &= \mathrm{diag}(q, p_m) + \mathrm{diag}(\mathbb{1}_{2n} - q, \mathbb{1}_{k-2n} - p_m) \\ &= \mathbb{1}_k \end{aligned}$$

So $z \in U_k(A^+)$ (It is not clear yet, that $z = \mathbb{1}_k \mod M(A)$.) . We have

$$\rho(z) = \mathrm{diag}(u, p_m) + \mathrm{diag}(0, x') = \mathrm{diag}(u, x'') \in U_k((A/J)^+) \ ,$$

i.e. $x'' \in U_{k-n}(\mathbb{C})$. Finally consider $z' := \mathrm{diag}(\mathbb{1}_n, (x'')^*)z$, then

$$\begin{aligned} \rho(z') = \rho(\mathrm{diag}(\mathbb{1}_n, (x'')^*)z) &= \rho(\mathrm{diag}(\mathbb{1}_n, (x'')^*))\rho(z) \\ \mathrm{diag}(\mathbb{1}_n, (x'')^*)\mathrm{diag}(u, x'') &= \mathrm{diag}(u, \mathbb{1}_{k_n}) \ . \end{aligned}$$

and since $u \in U_n(A/J)$ already, it holds that $z' \equiv \mathbb{1}_k \mod M_k(A)$, (Again, using that $\rho \equiv \rho^+$ does not act on the $\mathbb{C}$ component.) such that $z' \in U_k(A)$. So we have:

$$\rho_*([z']_1) = [\rho(z')]_1 = [\mathrm{diag}(u, \mathbb{1})]_1 = [u]_1 \ .$$

This shows that $\mathrm{Ker}(\partial) \subseteq \mathrm{Im}(\rho_*)$ and thus $\mathrm{Ker}(\partial) = \mathrm{Im}(\rho_*)$.

Exactness in $K_0(J)$: In remark 3.5.70, we have seen that $\iota_* \circ \partial = 0$, so $\mathrm{Im}(\partial) \subseteq \mathrm{Ker}(\iota_*)$.

On the other hand, let $[p]_{00} - [p_n]_{00} \in K_0(J)$, where $p \in M_{2n}(J^+)$ with $p - p_n \in M_{2n}(J)$ (see theorem 3.4.40) and $[p]_{00} - [p_n]_{00} = 0$ in $K_0(A)$. Then (again theorem 3.4.40) there is an $m \in \mathbb{N}$ and $k \geq m + 2n$, such that

$$\mathrm{diag}(p_n, p_m) \sim_u \mathrm{diag}(p, p_m) \ .$$

Using conjugation by unitary matrices, we can reorder the ones and zeros of $\mathrm{diag}(p_n, p_m)$, such that

$$p_{n+m} \sim_u \mathrm{diag}(p_n, p_m) \sim_u \mathrm{diag}(p, p_m) \ .$$

That is, there is a $u \in U_k(A^+)$, such that

$$up_{n+m}u^* = \mathrm{diag}(p, p_m) \ .$$

However, by adding zeros, i.e. increasing k, we can consider something of the form $\mathrm{diag}(u, u^*)$ instead, as the parts after u vanish because of the zeros:

$$\begin{aligned}\mathrm{diag}(u, \mathbb{1}_m, u^*, \mathbb{1}_m, \mathbb{1}_{2n-2m})\mathrm{diag}(p_{n+m}, 0)\mathrm{diag}(u^*, \mathbb{1}_m, u, \mathbb{1}_m, \mathbb{1}_{2n-2m})\\ = \mathrm{diag}(p, p_m, 0) \ .\end{aligned}$$

To simplify notation, we identify, by increasing k and n to $n + m$:

$$\begin{aligned}p_n &\equiv \mathrm{diag}(p_{n+m}, 0) \ ,\\ p &\equiv \mathrm{diag}(p, p_m, 0) \ ,\\ u &\equiv \mathrm{diag}(u, u^*, \mathbb{1}_{2n}) \in U_{4n}(A)_0 \ .\end{aligned}$$[7]

So Œ there is a $u \in U_{4n}(A)_0$, such that $up_nu^* = p$. It follows that

$$\rho(u)p_n = \rho(up_n) = \rho(up_nu^*u) = \rho(pu) = \rho(p)\rho(u) = p_n\rho(u) \ .$$

Thus it holds that $\rho(u) = \mathrm{diag}(u_1, u_2)$ with $u_1 \in U_n({}^{A}\!/\!{}_{J})$ and $u_2 \in U_{3n}({}^{A}\!/\!{}_{J})$. Because of $u \in U_{4n}(A)_0$, there is a path u_t from $u_0 = \mathbb{1}_{4n}$ to $u_1 = u$. Since ρ is unital, surjective and continuous, it holds that $\rho(U_{4n}(A)_0) = U_{4n}({}^{A}\!/\!{}_{J})_0$ (see corollary 3.2.26), i.e. $\rho(u_t)$ is a path from $\mathbb{1}_{4n}$ to $\rho(u_1) = \rho(u) = \mathrm{diag}(u_1, u_2)$. From corollary 3.5.47 we know, that by increasing n, we can assume that

$$u_2 \sim_h \mathrm{diag}(u_1^*, \mathbb{1}_{2n}) \ .$$

Multiplying from the left with $u_2^{-1} = u_2^*$ yields

$$\mathbb{1}_{3n} \sim_h u_2^*\mathrm{diag}(u_1^*, \mathbb{1}_{2n})$$

or put differently $u_2^*\mathrm{diag}(u_1^*, \mathbb{1}_{2n}) \in U_{3n}({}^{A}\!/\!{}_{J})_0$. Corollary 3.2.26 now assures the existence of a unitary lift, i.e. there is a $v \in U_{3n}(A)_0$, such that $\rho(v) = u_2^*\mathrm{diag}(u_1^*, \mathbb{1}_{2n})$. Define $w := u\mathrm{diag}(\mathbb{1}_n, v)$ to obtain:

$$\begin{aligned}\rho(w) &= \mathrm{diag}(u_1, u_2)\mathrm{diag}(\mathbb{1}_n, u_2^*\mathrm{diag}(u_1^*, \mathbb{1}_{2n}))\\ &= \mathrm{diag}(u_1, u_2)\mathrm{diag}(\mathbb{1}_n, u_2^*)\mathrm{diag}(\mathbb{1}_n, u_1^*, \mathbb{1}_{2n})\\ &= \mathrm{diag}(u_1, u_1^*, \mathbb{1}_{2n}) \ .\end{aligned}$$

and because $p_n = \mathrm{diag}(\mathbb{1}_n, 0_{3n})$ in $M_{4n}(A)$:

$$p = up_nu^* = u\mathrm{diag}(\mathbb{1}_n, v)p_n\mathrm{diag}(\mathbb{1}_n, v^*)u^* = wp_nw^* \ .$$

Hence it follows that

$$\partial([\mathrm{diag}(u_1, u_1^*, \mathbb{1}_{2n})]_1) = [wp_nw^*]_{00} - [p_n]_{00} = [p]_{00} - [p_n]_{00} \ ,$$

i.e. $\mathrm{Ker}(\iota_*) \subseteq \mathrm{Im}(\partial)$ and thus $\mathrm{Ker}(\iota_*) = \mathrm{Im}(\partial)$. □

[7] Again, here we use corollary 3.3.35 to find that $\mathrm{diag}(u, \mathbb{1}_m, u^*, \mathbb{1}_m, \mathbb{1}_{2n-2m}) \sim_h \mathbb{1}_{4n}$, so $\mathrm{diag}(u, \mathbb{1}_m, u^*, \mathbb{1}_m, \mathbb{1}_{2n-2m}) \in U_{4n}(A)_0$.

Corollary 3.5.77.
Because of lemma 3.5.55 it holds that

$$K_{n+1}(J) \xrightarrow{\iota_*} K_{n+1}(A) \xrightarrow{\rho_*} K_{n+1}(A/J) \xrightarrow{\partial} K_n(J) \xrightarrow{\iota_*} K_n(A) \xrightarrow{\rho_*} K_n(A/J)$$

is an exact sequence.

Proof 3.5.78.
It is enough to use that $K_1(S^n(J)) \cong K_{n+1}(J)$ and $K_0(S^n(J)) \cong K_n(J)$ etc. because of theorem 3.5.45. □

Definition 3.5.79.
The short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called **splitting**, if there is a morphism $h\colon C \to B$, such that $g \circ h = \mathrm{Id}_C$. The right inverse morphism h is also called **section**.

The splitting lemma allows to characterize splitting short exact sequences by different conditions.

Lemma 3.5.80 (**Splitting lemma**).
Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence, then the following claims are equivalent:

i) The short exact sequence is splitting.

ii) There exists a morphism $r\colon B \to A$, such that $r \circ f = \mathrm{Id}_A$.

iii) There is an isomorphism $I\colon B \to A \oplus C$, such that $I \circ f\colon A \to A \oplus C$ is the canonical inclusion and $g \circ I^{-1}\colon A \oplus C \to C$ is the canonical projection.

The proof can be found on [Wik19], for example.

Lemma 3.5.81.
Let the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} A/J \longrightarrow 0$$

be splitting. Then the sequence

$$0 \longrightarrow K_n(J) \xrightarrow{\iota_*} K_n(A) \underset{\sigma_+}{\overset{\rho_*}{\rightleftarrows}} K_n(A/J) \longrightarrow 0$$

is a splitting short exact sequence for all n.

Proof 3.5.82.
Because of functoriality, it holds that $S\rho \circ S\sigma = \mathrm{Id}_{S(A/J)}$. Together with lemma 3.5.55, this means, that suspension preserves the split exactness.

We obtain the following sequence

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \underset{\sigma_*}{\overset{\rho_*}{\rightleftarrows}} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{\iota_*} K_0(A) \underset{\sigma_*}{\overset{\rho_*}{\rightleftarrows}} K_0(A/J)$$

that is exact because of theorem 3.5.75. Again, because of functoriality, it holds that $\rho_* \circ \sigma_* = \mathrm{Id}_{K_0(A/J)}$ and $\rho_* \circ \sigma_* = \mathrm{Id}_{K_1(A/J)}$. Then, for all $x \in K_{0,1}(A/J)$, it holds that $\rho_*(\sigma_*(x)) = x$, so σ_* is surjective. Hence, the sequence is exact in $K_0(A/J)$, i.e. we can add a zero on the right:

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \underset{\sigma_*}{\overset{\rho_*}{\rightleftarrows}} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{\iota_*} K_0(A) \underset{\sigma_*}{\overset{\rho_*}{\rightleftarrows}} K_0(A/J) \longrightarrow 0$$

For the same reason, $\rho_*\colon K_1(A) \to K_1(A/J)$ is surjective, i.e. $\mathrm{Im}(\rho_*) = K_1(A/J)$. Because of theorem 3.5.75, it holds that

$$\mathrm{Ker}(\partial) = \mathrm{Im}(\rho_*) = K_1(A/J) \ ,$$

$$\partial\colon K_1(A/J) \longrightarrow K_0(J) \ , \quad [u]_1 \longmapsto 0 \ .$$

This means, that ∂ is the zero map, and we can replace it by $0 \to$ in the sequence, to obtain the following short exact sequence that splits.

$$0 \longrightarrow K_0(J) \xrightarrow{\iota_*} K_0(A) \underset{\sigma_*}{\overset{\rho_*}{\rightleftarrows}} K_0(A/J) \longrightarrow 0$$

Since suspension preserves the exactness and the splitting property, the proof holds for general n (using corollary 3.5.77 and $K_n(A) \cong K_0(S^n A)$). □

Corollary 3.5.83.
Let the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} A/J \longrightarrow 0$$

be splitting, then it holds that $K_n(A) \cong K_n(J) \oplus K_n(A/J)$.

Proof 3.5.84.
By lemma 3.5.81, the sequence

$$0 \longrightarrow K_n(J) \xrightarrow{\iota_*} K_n(A) \underset{\sigma_+}{\overset{\rho_*}{\rightleftarrows}} K_n(A/J) \longrightarrow 0$$

short exact and splitting. The rest follows from the splitting lemma. □

Lemma 3.5.85.
The inclusion $A \subseteq A^+$ induces an isomorphism of abelian groups $K_1(A) \cong K_1(A^+)$.

Proof 3.5.86.
The short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} A^+ \underset{j}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

splits for the map $j\colon \mathbb{C} \to A^+$, $z \mapsto (0, z)$. Furthermore A is a closed ideal of $A^+ \cong A \times \mathbb{C}$ and $A^+/A \cong \mathbb{C}$, as seen on page 127. So by lemma 3.5.81, the following sequence is short exact and splitting:

$$0 \longrightarrow K_1(A) \xrightarrow{\iota_*} K_1(A^+) \underset{j_*}{\overset{\pi_*}{\rightleftarrows}} K_1(\mathbb{C}) \longrightarrow 0$$

With more topology, it can be shown, that $V(S\mathbb{C}) = 0$ and thus $K_0(S\mathbb{C}) = 0$. Because of theorem 3.5.45 it holds that

$$K_1(\mathbb{C}) = K_0(S\mathbb{C}) = 0 \ ,$$

so that π_* is the zero map. From corollary 3.5.83 it follows that

$$K_1(A^+) \cong K_1(A) \oplus K_1(\mathbb{C}) = K_1(A) \oplus 0 \cong K_1(A) \ .$$

□

So far, the results of exact sequences only hold for ideals and quotients. However, identifying short exact sequences with ideals and quotients allows to carry over the results for all short exact sequences.

Lemma 3.5.87.
Let $\phi\colon A \to B$ be a $*$-morphism of local C^*-algebras and

$$0 \longrightarrow \mathrm{Ker}(\phi) \xrightarrow{\iota} A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence. Then $J := \mathrm{Ker}(\phi)$ is a closed ideal of A and $B \cong A/J$.

Proof 3.5.88.
As $*$-morphism, ϕ is continuous, by corollary 2.6.14. Hence $J = \mathrm{Ker}(\phi) = \phi^{-1}(0)$ is closed. To see that J is indeed an ideal, one calculates for $a \in A$, $b \in J$:

$$\phi(ab) = \phi(a)\phi(b) = \phi(a) \cdot 0 = 0 \quad ab \in J \quad \text{etc.} \ .$$

From the short exactness, it follows that ϕ is surjective, i.e. $B = \mathrm{Im}(\phi)$. Because of the isomorphism theorem, it holds that

$$B = \mathrm{Im}(\phi) \cong {}^{A}\!/_{\mathrm{Ker}(\phi)} = {}^{A}\!/_{J} \ .$$

□

3.6 Preparations for the Bott-periodicity

However, proving this theorem is no easy task, requiring further concepts from C^*-algebra theory, not yet introduced. In the following we will recap the results form [All17, Section 2.3, 2.4 and3.6], mostly without proofs.

3.6.1 C^*-algebras by generators and relations

Definition 3.6.1.
Let $\mathcal{G} = (x_j)_{j\in J}$ be a set of generators. The **free $*$-algebra** defined by $\mathcal{G}$ is the vector space $\mathcal{F}\langle\mathcal{G}\rangle = \mathcal{F}\langle x_j \mid j \in J\rangle$ with a basis, consisting of all non-empty words from the alphabet $(x_j, x_j^*)_{j\in J}$. The Multiplication is defined as bilinear extension of the composition of two words to a new word

$$(w_1, w_2) \longmapsto w_1 w_2 = w_3$$

and the $*$-map is the antilinear extension of the anti-involution $x_j \mapsto x_j^*$.

By this definition, there is no unit element, since empty words are not allowed. Hence, we include the $\mathbb{1}$ by hand: $\mathbb{C}\mathbb{1}\oplus\mathcal{F}\langle\mathcal{G}\rangle$. Elements of $\mathbb{C}\mathbb{1}\oplus\mathcal{F}\langle\mathcal{G}\rangle$ are called **non-commutative $*$-polynomials**.

$$p \equiv p(x_{j_1}, \dots, x_{j_n}, x_{j_1}^*, \dots, x_{j_n}^*) \in \mathbb{C}\mathbb{1} \oplus \mathcal{F}\langle\mathcal{G}\rangle \ .$$

Definition 3.6.2.
A **$*$-relation** on the generators $\mathcal{G} = (x_j)_{j\in J}$ is a pair (p, η), where $p \in \mathbb{C}\mathbb{1}\oplus\mathcal{F}\langle\mathcal{G}\rangle$ and $\eta \in \mathbb{R}_{\geq 0}$. Let now $\mathcal{R}$ denote a set of relations on $\mathcal{G}$. A **representation** of $(\mathcal{G}, \mathcal{R})$ is a pair $(\mathcal{H}, \rho)$, where $\mathcal{H}$ is a Hilbert space and ρ a map $\rho\colon \mathcal{G} \to \mathcal{L}(\mathcal{H})$, $x_j \mapsto A_j$, such that

$$\|p(A_{j_1}, \dots, A_{j_n}, A_{j_1}^*, \dots, A_{j_n}^*)\| \leq \eta \ .$$

Remark 3.6.3.
The relations can also be of the form $p = 0$, which becomes an algebraic relation. Consider for example $u^*u - \mathbb{1} = 0$, which reads $u^*u = \mathbb{1}$, etc. In this case, it immediately follows that if $p = 0$, it holds that $\|p\| = 0$.

Every representation of $(\mathcal{H}, \rho)$ of $(\mathcal{G}, \mathcal{R})$ can be uniquely extended to a $*$-morphism $\widetilde{\rho}\colon \mathcal{F}\langle\mathcal{G}\rangle \to \mathcal{L}(\mathcal{H})$ as follows. Let $x_{i_1} \dots x_{i_m} x_{j_1}^* \dots x_{j_n}^* \dots$ be a general word from the alphabet $\mathcal{G}$, which is a basis element of $\mathcal{F}\langle\mathcal{G}\rangle$. Then $\widetilde{\rho}$ is defined (linearly) by

$$\widetilde{\rho}(x_{i_1} \dots x_{i_m} x_{j_1}^* \dots x_{j_n}^* \dots) = A_{i_1} \dots A_{i_m} A_{j_1}^* \dots A_{j_n}^* \dots \ .$$

Definition 3.6.4.
A pair $(\mathcal{G}, \mathcal{R})$ of generators and relations is called **admissible**, if for every family

$(\mathcal{H}_i, \rho_i)_{i\in I}$ of representations, and every $x \in \mathcal{G}$, it holds that

$$\bigoplus_{i\in I} \rho_i(x) \in \mathcal{L}\left(\bigoplus_{i\in I} \mathcal{H}_i\right) .$$

If $(\mathcal{G}, \mathcal{R})$ is admissible, then $(\bigoplus_{i\in I} \mathcal{H}_i, \bigoplus_{i\in I} \rho_i)$ is a representation.

Theorem 3.6.5.
Let $(\mathcal{G}, \mathcal{R})$ be an admissible pair of generators and relations. Then

$$\|p\| := \sup\{\|\widetilde{\rho}(p)\| \mid (\mathcal{H}, \rho) \text{ are representations of } (\mathcal{G}, \mathcal{R})\} .$$

defines a C^*-semi-norm on $\mathcal{F}\langle\mathcal{G}\rangle$.

Now, if $(\mathcal{G}, \mathcal{R})$ is an admissible pair of generators and relations, we can define $C^*(\mathcal{G}, \mathcal{R})$ as the completion of

$$\mathcal{F}\langle\mathcal{G}\rangle / \{\|\cdot\|=0\} ,$$

where $\|\cdot\|$ is the semi-norm from the theorem. The C^*-algebra $C^*(\mathcal{G}, \mathcal{R})$ is called the **universal C^*-algebra** of $(\mathcal{G}, \mathcal{R})$. It can be shown, that the universal C^*-algebra has the following universal property: If there is a representation $(\mathcal{H}, \rho)$ of $(\mathcal{G}, \mathcal{R})$, there exists a unique $*$-representation $(\mathcal{H}, \rho')$ of $C^*(\mathcal{G}, \mathcal{R})$, such that the following diagram of $*$-morphisms commutes:

$$\begin{array}{ccc} C^*(\mathcal{G}, \mathcal{R}) & \xrightarrow{\rho'} & \mathcal{L}(\mathcal{H}) \\ \uparrow & \nearrow_{\widetilde{\rho}} & \\ \mathcal{F}\langle\mathcal{G}\rangle & & \end{array}$$

where $\mathcal{F}\langle\mathcal{G}\rangle \to C^*(\mathcal{G}, \mathcal{R})$ is the inclusion w.r.t. the construction of $C^*(\mathcal{G}, \mathcal{R})$.

Theorem 3.6.6.
Let A be a Banach-$*$-algebra, $\mathcal{G} = A$ and $\mathcal{R} = \{(p, 0)\}$. Then $C^*(A) := C^*(\mathcal{G}, \mathcal{R})$ is called the ***enveloping C^*-algebra*** *of A and satisfies the following universal property: There is a canonical $*$-morphism $A \to C^*(A)$ and for every $*$-representation $(\mathcal{H}, \rho)$ of A, there exists a unique $*$-representation $(\mathcal{H}, \rho')$ of $C^*(A)$, such that the following diagram commutes:*

$$\begin{array}{ccc} C^*(A) & \xrightarrow{\rho'} & \mathcal{L}(\mathcal{H}) \\ \uparrow & \nearrow_{\rho} & \\ A & & \end{array}$$

3.6.2 Toeplitz algebras

We consider $\ell^2(\mathbb{Z})$, the set of sequences $f = (f_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ with

$$\|f\|_2^2 := \sum_{n\in\mathbb{Z}} |f_n|^2 < \infty .$$

Then $\ell^2(\mathbb{Z})$ is a separable Hilbert space with the Hilbert basis $\{e_n\}_{n\in\mathbb{Z}}$, where

$$(e_n)_m = \delta_{nm} \ , \quad \forall\, n,m \in \mathbb{Z} \ .$$

The closed subspace $\ell^2(\mathbb{N}_0)$ is the closed span of the $\{e_n\}_{n\in\mathbb{N}_0}$.

Definition 3.6.7.
The **shift operator** S on $\ell^2(\mathbb{N}_0)$ is the unitary operator, defined by

$$Se_n = e_{n+1} \ .$$

Let $Q\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N}_0)$ be the orthogonal projection, then the **unilateral shift operator** is $\widehat{S} \coloneqq QSQ$.

The unilateral shift operator acts as follows:

$$\widehat{S}e_n = \begin{cases} e_{n+1} & , \ n \geq 0 \\ 0 & , \ \text{else.} \end{cases} \ .$$

Consider the Fourier transformation

$$\mathcal{F}\colon \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{S}^1) \ , \quad \mathcal{F}(f)(z) \coloneqq \sum_{n\in\mathbb{Z}} f_n z^n$$

The integration measure on $L^2(\mathbb{S}^1)$ is given by the pushforward measure of the Lebesgue measure λ w.r.t. $[-\pi,\pi] \ni k \mapsto e^{ik}$:

$$\int_{\mathbb{S}^1} \psi(z)\ dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(e^{ik})\ dk = \frac{1}{2\pi i}\oint \psi(z)z^{-1}\ dz \ .$$

In the last step, we used the definition of curve integrals

$$\int_\gamma f(z)\ dz = \int_I f(\gamma(k))\gamma'(k)\ dk$$

for $\gamma\colon [-\pi,\pi] \to \mathbb{C}$, $k \mapsto e^{ik}$:

$$\begin{aligned} \frac{1}{2\pi i}\oint \psi(z)z^{-1}\ dz &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \psi(e^{ik})e^{-ik}\frac{d}{dk}e^{ik}\ dk \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \psi(e^{ik})e^{-ik} i e^{ik}\ dk \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(e^{ik})\ dk \end{aligned}$$

Since $\mathbb{S}^1 \ni z = e^{i\varphi}$ for any $\varphi \in [-\pi,\pi]$, it holds that

$$\overline{z^m} = \overline{(e^{i\varphi})^m} = \overline{e^{im\varphi}} = e^{-im\varphi} = (e^{i\varphi})^{-m} = z^{-m} \ .$$

So one obtains:

$$\langle z^m \mid z^n\rangle = \int_{\mathbb{S}^1} z^{n-m}\ dz = \frac{1}{2\pi i}\oint_{|z|=1} z^{n-m-1} = \delta_{mn} \ .$$

By the Stone-Weierstrass theorem, it holds that $\mathbb{C}[z, z^{-1}] = \langle z^n \mid n \in \mathbb{Z}\rangle_{\mathbb{C}}$ is dense in $C(\mathbb{S}^1)$. Hence $\{z^n\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2(\mathbb{S}^1)$. It especially holds that the series in the definition of $\mathcal{F}(f)$ converges (by definition of $\ell^2(\mathbb{Z})$):

$$\begin{aligned}\|\mathcal{F}(f)(z)\|^2 &= \left\langle \sum_{n\in\mathbb{Z}} f_n z^n \mid \sum_{m\in\mathbb{Z}} f_m z^m \right\rangle = \sum_{m,n\in\mathbb{Z}} \overline{f_n} f_m \langle z_n \mid z_m\rangle \\ &= \sum_{m,n\in\mathbb{Z}} \delta_{mn} = \sum_{n\in\mathbb{Z}} \overline{f_n} f_n = \sum_{n\in\mathbb{Z}} |f_n|^2 \ .\end{aligned}$$

We also observer that $\mathcal{F}$ defines a unitary isomorphism.

$$\mathcal{F}^*(\psi)_n = \langle e_n \mid \mathcal{F}^*(\psi)\rangle_{\ell^2(\mathbb{Z})} = \langle \mathcal{F}(e_n) \mid \psi\rangle_{L^2(\mathbb{S}^1)} = \int_{\mathbb{S}^1} z^{-n}\psi(z)\ dz \ .$$

So especially it follows that

$$\mathcal{F}^*(z^m)_n = \langle e_n \mid \mathcal{F}^*(z^m)\rangle_{\ell^2(\mathbb{Z})} = \langle \mathcal{F}(e_n) \mid z^m\rangle_{L^2(\mathbb{S}^1)} = \langle z^n \mid z^m\rangle_{L^2(\mathbb{S}^1)} = \delta_{mn} \ ,$$

which shows that $\mathcal{F}^* = \mathcal{F}^{-1}$.

The image under $\mathcal{F}$ of $\ell^2(\mathbb{N}_0)$ is called **Hardy space** H^2 and is given by

$$H^2 = \mathcal{F}(\ell^2(\mathbb{N}_0)) = \overline{\langle z^n \mid n \in \mathbb{N}_0\rangle_{\mathbb{C}}} \ .$$

Remark 3.6.8.
Recall that $L^\infty(\mathbb{S}^1)$ is defined with respect to $\|f\|_{L^\infty} :=$ ess-sup$|f|$, where the essential supremum is given by

$$\text{ess-sup} f = \inf_{\substack{N\subset\mathbb{S}^1 \\ \lambda(N)=0}} \left(\sup_{\mathbb{S}^1/N} f\right) \ .$$

Definition 3.6.9.
Let P be the projection $: L^2(\mathbb{S}^1) \to H^2$ and $f \in L^\infty(\mathbb{S}^1)$. The **Toeplitz operator** $T_f \in \mathcal{L}(H^2)$ is defined as $T_f(h) := P(fh)$ for all $h \in H^2$.

It follows that

$$\begin{aligned}T_z(h) = P(zh) = P\left(z\sum_{n\in\mathbb{N}_0} h_n z^n\right) &= P\left(\sum_{n\in\mathbb{N}_0} h_n z^{n+1}\right) \\ &= \sum_{n\in\mathbb{N}_0} h_n z^{n+1} = \mathcal{F}\left(\sum_{n\in\mathbb{N}_0} h_n e_{n+1}\right) = (\mathcal{F}\circ\widehat{S})\left(\sum_{n\in\mathbb{N}_0} h_n e_n\right) \\ &= (\mathcal{F}\circ\widehat{S}\circ\mathcal{F}^*)(h)\end{aligned}$$

$$\Rightarrow \quad T_z = \mathcal{F}\circ\widehat{S}\circ\mathcal{F}^* \ .$$

Lemma 3.6.10.

Let $f \in L^\infty(\mathbb{S}^1)$, then it holds that $T_f^* = T_{\overline{f}}$ and

$$\|f\|_{L^\infty} = \|T_f\| = \|T_f\|_{\mathbb{L}/\mathcal{K}} := \inf\{\|T_f + K\| \mid K \in \mathcal{K}(H^2)\} .$$

Lemma 3.6.11.
Let $f \in L^\infty(\mathbb{S}^1)$, then $[T_z, T_f]$ has rank less than one.

Theorem 3.6.12.
Let $f \in C(\mathbb{S}^1)$ and $g \in L^\infty(\mathbb{S}^1)$, then the following operators are compact:

$$T_f T_g - T_{fg} , \qquad T_{fg} - T_g T_f , \qquad [T_f, T_g] .$$

Theorem 3.6.13.
*For the **Toeplitz algebra** $\mathcal{T} := C^*(T_z)$ it holds that*

$$\mathcal{T} = \{T_f + K \mid f \in C(\mathbb{S}^1),\ K \in \mathcal{K}(H^2)\} =: \mathcal{T}_0 .$$

The map

$$\pi \colon \mathcal{T} \longrightarrow C(\mathbb{S}^1) , \quad \pi(T_f + K) := f$$

is a well defined $*$-morphism, inducing the following short exact sequence of $*$-morphisms:

$$0 \longrightarrow \mathcal{K}(H^2) \longrightarrow \mathcal{T} \xrightarrow{\ \pi\ } C(\mathbb{S}^1) \longrightarrow 0$$

The map $\sigma \colon C(\mathbb{S}^1) \to \mathcal{T}$, $f \mapsto T_f$ is an isometrical section as liner map. The algebra $\mathcal{T}$ is irreducible on H^2 and contains only $\mathcal{K}(H^2)$ as minimal closed ideal.

The short exact sequence splits in the vector space sense. Since σ is not a $*$-morphism, the short exact sequence is not splitting in the C^*-algebra sense.

Lemma 3.6.14 (Wold-decomposition).
Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{L}(\mathcal{H})$ an isometry on $\mathcal{H}$, i.e. $x^*x = \mathbb{1}$. Then there is a Hilbert space $\mathcal{H}'$, $u \in U(\mathcal{H}')$ and a set I, such that x is unitarily equivalent to

$$T_z^{(I)} \oplus u := \bigoplus_I T_z \oplus u .$$

Theorem 3.6.15 (Coburn).
The Toeplitz algebra $\mathcal{T}$, together with the $*$-morphism $\mathcal{F}\langle u\rangle \to \mathcal{T}$, $u \mapsto T_z$ is the universal C^*-algebra for the generator $\mathcal{G} = \{u\}$ with relation $\mathcal{R} = \{u^*u = 1\}$.

3.6.3 Toeplitz extension

Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-sub algebra and $\mathcal{T}_A$ the C^*-sub algebra of $\mathcal{L}(\mathcal{H} \otimes H^2)$ generated by A and the Toeplitz operator $u = T_z$. Then $\mathcal{T}_A$ is the universal C^*-algebra of the generators u and $\{a \mid a \in A\}$ with relations[8] $u^* u = \mathbb{1}$, $au = ua$ for all $a \in A$ as well as the $*$-algebra relations of A. This follows from the Coburn theorem 3.6.15 and the properties of the tensor product, which allow for the universal property to hold.

The algebra $\Omega A := C(\mathbb{S}^1, A)$ is also a C^*-sub algebra of $\mathcal{L}(\mathcal{H} \otimes H^2)$. The map $a \otimes u \mapsto z \cdot a$ defines a unique surjective $*$-morphism $\pi_A \colon \mathcal{T}_A \to \Omega A$ (see theorem 3.6.13). The kernel $\mathrm{Ker}(\pi_A)$ of π_A is generated by A and $e := \mathbb{1} - uu^*$.[9]

Lemma 3.6.16.
The element e can be written as $e = \mathbb{1} \otimes |1\rangle\langle 1|$.

Proof 3.6.17.
Recall that $u = T_z \in \mathcal{L}(H^2)$, and that $\{|z^n\rangle\}_{n \in \mathbb{N}_0}$ is a Hilbert basis of the Hardy space. So by definition $u(z^n) = T_z(z^n) = z^{n+1}$. Since $z = e^{i\varphi} \in \mathbb{S}^1$, we have $\overline{z} z^n = z^{n-1}$, such that with lemma 3.6.10 and $P = \sum_{k \in \mathbb{N}_0} |z^k\rangle\langle z^k|$:

$$
\begin{aligned}
u^*(z^n) = T_z^*(z^n) = T_{\overline{z}}(z^n) = P(\overline{z} z^n) &= \sum_{k \in \mathbb{N}_0} |z^k\rangle\langle z^k \mid z^{n-1}\rangle \\
&= \sum_{k \in \mathbb{N}_0} |z^k\rangle \delta_{k,n-1} = |z^{n-1}\rangle \equiv z^{n-1} \, .
\end{aligned}
$$

Thus

$$
\begin{aligned}
e(z^n) = (\mathbb{1} - uu^*)(z^n) &= \begin{cases} z^n - u(z^{n-1}) & , \ n \geq 1 \\ z^n - u(0) & , \ n = 0 \end{cases} \\
&= \begin{cases} z^n - z^n = 0 & , \ n \geq 1 \\ z^0 = 1 & , \ n = 0 \end{cases} \, .
\end{aligned}
$$

On the other hand

$$
|1\rangle\langle 1|(z^n) = |z^0\rangle\langle z^0|(z^n) = |z^0\rangle\langle z^0 \mid z^n\rangle = \begin{cases} 0 & , \ n \geq 1 \\ 1 & , \ n = 0 \end{cases} \, .
$$

□

From theorem 3.6.13 it also follows that $\mathrm{Ker}(\pi_A) \cong A \otimes \mathcal{K}(H^2)$, since $\pi(K) = 0$ for $K \in \mathcal{K}(H^2)$.

[8] More carefully, we would need to write $a = a \otimes \mathbb{1}$ and $u = \mathbb{1} \otimes u$, such that it naturally follows that:

$$
au \equiv (a \otimes \mathbb{1}) \circ (\mathbb{1} \otimes u) = a \otimes u = (\mathbb{1} \otimes u)(a \otimes \mathbb{1}) \equiv ua \, .
$$

[9] With $z \in \mathbb{S}^1$ $\pi_A(a \otimes (\mathbb{1} - uu^*)) = (1 - zz^*)a = (1-1)a = 0 \cdot a = 0$.

Definition 3.6.18.
The **Toeplitz extension** is the following short exact sequence:

$$0 \longrightarrow A \otimes \mathcal{K}(H^2) \xrightarrow{\subset} \mathcal{T}_A \xrightarrow{\pi_A} \Omega A \longrightarrow 0 \ .$$

Let $q_A \colon \mathrm{ev}_1 \circ \pi_A \colon \mathcal{T}_A \to A$ and define $\mathcal{T}_{A,0} \coloneqq \mathrm{Ker}(q_A)$. Then $\mathcal{T}_{A,0}$ is generated by A and $\mathbb{1} - u$, so $\mathcal{T}_{A,0} \subset \mathcal{T}_A$. Indeed, $\mathrm{ev}_1(\pi_A(\mathbb{1} - u)) = \mathrm{ev}_1(1 - z) = 1 - 1 = 0$ and

$$(1 - u)(u^* - 1) + (1 - u) + (1 - u^*) = 1 - uu^* \ .$$

The following sequence is a short exact sequence:

$$0 \longrightarrow \mathcal{T}_{A,0} \xrightarrow{\subset} \mathcal{T}_A \xrightarrow{q_A} A \longrightarrow 0 \ .$$

This short exact sequence splits with $\sigma_A \colon A \to \mathcal{T}_A \subset L^2(\mathcal{H} \otimes H^2)$, defined by[10]

$$(\sigma_A(a)\psi)(z) \coloneqq a(\psi(z)) \ , \quad \forall\, a \in A \ , \ \psi \in \mathcal{H} \otimes H^2 \ , \ z \in \mathbb{S}^1 \ .$$

For $v \otimes f \in \mathcal{H} \otimes H^2$ it holds that $\sigma_A(a)(v \otimes f) = (av) \otimes f$, such that we can write $\sigma_A(a) = a \otimes \mathbb{1}$. As claimed, it holds that

$$q_A(\sigma_A(a)) = q_A(a \otimes \mathbb{1}) = \mathrm{ev}_1(\pi_A(a \otimes \mathbb{1})) = \mathrm{ev}_1(1 \cdot a) = a \ .$$

Corollary 3.6.19.
It holds that $K_\bullet(\mathcal{T}_{A,0}) \cong \mathrm{Ker}((q_A)_*)$.

Proof 3.6.20.
From lemma 3.5.81 (applying lemma 3.5.87) we know that

$$0 \longrightarrow K_\bullet(\mathcal{T}_{A,0}) \xrightarrow{\subset} K_\bullet(\mathcal{T}_A) \xrightarrow{(q_A)_*} K_\bullet(A) \longrightarrow 0 \ .$$

is also short exact. But then, short exactness means that

$$K_\bullet(\mathcal{T}_{A,0}) = \mathrm{Im}(\subset) = \mathrm{Ker}((q_A)_*) \ .$$

□

Lemma 3.6.21.
Let $s \in U(A)$ *be self adjoint, then it holds that* $s \in U(A)_0$.

[10] Recall that $\mathcal{H} \otimes H^2 = \mathcal{H} \otimes L^2(\mathbb{S}^1)$. So for $v \otimes f \in \mathcal{H} \otimes H^2$ and $z \in \mathbb{S}^1$, we have $(v \otimes f)(z) = f(z)v$. Hence we can define for $\psi \in \mathcal{H} \otimes H^2$ the element $\psi(z) \in \mathcal{H}$.

Proof 3.6.22.
Let $p \coloneqq \frac{1}{2}(\mathbb{1} + s)$, so $s = 2p - \mathbb{1}$. Also note, that p is also self adjoint. We define

$$u_t \coloneqq p + e^{i\pi(t-1)}(\mathbb{1} - p) \quad \text{for } t \in [0,1] \ .$$

Since $s^* = s$ and $s \in U(A)$, i.e. $ss^* = s^*s = \mathbb{1}$, it holds that $s^2 = \mathbb{1}$, such that we find

$$p^2 = \frac{1}{4}(\mathbb{1} + 2s + s^2) = \frac{1}{2}(\mathbb{1} + s) = p \ .$$

Then it holds that $p - p^2 = 0$ and thus

$$\begin{aligned} u_t^* u_t &= (p + e^{-i\pi(t-1)}(\mathbb{1} - p))(p + e^{i\pi(t-1)}(\mathbb{1} - p)) \\ &= p^2 + e^{-i\pi(t-1)} e^{i\pi(t-1)}(\mathbb{1} - p)^2 = p^2 + (\mathbb{1} - p)^2 \\ &= p^2 + \mathbb{1} - 2p + p^2 = p + \mathbb{1} - 2p + p = \mathbb{1} \\ &= \ldots = u_t u_t^* \ . \end{aligned}$$

This means that u_t is unitary. Since

$$u_0 = p + e^{-i\pi}(\mathbb{1} - p) = p - (\mathbb{1} - p) = 2p - \mathbb{1} = s$$

$$\text{and} \quad u_1 = p + e^0(\mathbb{1} - p) = p + (\mathbb{1} - p) = \mathbb{1}$$

the claim follows. □

Theorem 3.6.23.
The map $(q_A)_* \colon K_\bullet(\mathcal{T}_A) \to K_\bullet(A)$ is invertible with inverse $(\sigma_A)_*$.

Corollary 3.6.24.
It holds that $K_\bullet(\mathcal{T}_{A,0}) = 0$.

Proof 3.6.25.
In corollary 3.6.19, we have seen that $K_\bullet(\mathcal{T}_{A,0}) \cong \operatorname{Ker}((q_A)_*)$. However, we have shown that $(q_A)_*$ is an isomorphism, i.e. injective, so $\operatorname{Ker}((q_A)_*) = 0$. □

3.7 Bott-Periodicity

The Bott-periodicity, based on the equally called result from topology, is one of the central results in K-theory of C^*-algebras, stating that $K_1(SA)$ and $K_0(A)$ are isomorphic. Here we will prove the central result of K-theory in two different ways.

Theorem 3.7.1 (**Bott-periodicity**)**.**
Let A be a local C^*-algebra. Then there is a natural isomorphism of abelian groups $K_1(SA) \to K_0(A)$.

Remark 3.7.2.
The meaning of naturality is the same as for natural transformations here. Let $\phi\colon A \to B$ be a $*$-morphism, then **natural** means, that the following diagram commutes:

$$\begin{array}{ccc} K_1(SA) & \xrightarrow{\partial} & K_0(A) \\ {\scriptstyle K_1(\phi)}\downarrow & & \downarrow{\scriptstyle K_0(\phi)} \\ K_1(SB) & \xrightarrow[\partial]{} & K_0(B) \end{array}$$

Remark 3.7.3.
Here, we show the proof for the C^*-algebra case. With some effort and corollary 3.2.6, it can be shown, that the local C^*-algebra case can be reduced to the C^*-algebra case.

Proof 3.7.4 (**Only for C^*-algebras**).
Restricting the short exact sequence from definition 3.6.18 to $\mathcal{T}_{A,0}$ we obtain the following short exact sequence:

$$0 \longrightarrow A \otimes \mathcal{K}(H^2) \longrightarrow \mathcal{T}_{A,0} \xrightarrow{\pi_A} SA \longrightarrow 0$$

Recall, that a and $\mathbb{1} - u$ generate $\mathcal{T}_{A,0} = \operatorname{Ker}(q_A)$. It holds that $\pi_A(a \otimes (\mathbb{1} - u)) = (1 - z)a \in SA$, as $(1 - 1)a = 0$ for $1 \in \mathbb{S}^1$. Using the identification from lemma 3.5.87 allows to apply the long exact sequence of K-theory (theorem 3.5.75). We obtain:

$$K_1(\mathcal{T}_{A,0}) \xrightarrow{(\pi_A)_*} K_1(SA) \xrightarrow{\partial} K_0(A \otimes \mathcal{K}(H^2)) \longrightarrow K_0(\mathcal{T}_{A,0})$$

It holds that the outer groups $K_1(\mathcal{T}_{A,0}) = K_0(\mathcal{T}_{A,0}) = 0$, such that ∂ is a bijection, i.e. an isomorphism $\partial\colon K_1(SA) \xrightarrow{\cong} K_0(A \otimes \mathcal{K}(H^2))$. From lemma 3.2.21 and corollary 3.4.45 it follows that $K_0(A \otimes \mathcal{K}(H^2)) \cong K_0(A)$, and thus:

$$\sigma^{-1} \circ \partial\colon K_1(SA) \xrightarrow{\cong} K_0(A)\ .$$

□

Remark 3.7.5.
It is common practice to write $\partial \equiv \sigma^{-1} \circ \partial\colon K_1(SA) \xrightarrow{\cong} K_0(A)$, hiding the isomorphy $K_\bullet(A) \cong K_\bullet(A \otimes \mathcal{K})$.

Corollary 3.7.6.

There are natural isomorphisms

$$K_n(A) \cong \begin{cases} K_0(A) & , \textit{for even } n \\ K_1(A) & , \textit{for odd } n \end{cases} .$$

Proof 3.7.7.
From theorem 3.5.45 we know that $K_1(A) \cong K_0(SA)$ and by definition 3.5.64, $K_n(A) \cong K_{n-1}(SA) \cong \ldots \cong K_0(S^nA)$. On the other hand, the Bott-periodicity states that $K_1(SA) \cong K_0(A)$.

Indeed, for $K_2(A)$ we find

$$K_2(A) \cong K_1(SA) \cong K_0(A)$$

$$\text{and} \quad K_3(A) \cong K_1(S^2A) \cong K_0(SA) \cong K_1(A) .$$

So we can assume that $K_{2m}(A) \cong K_0(A)$ and $K_{2m+1}(A) \cong K_1(A)$ for $m \in \mathbb{N}_0$. Completing the induction, we calculate:

$$\begin{aligned} K_{2(m+1)}(A) &= K_{2m+2}(A) \cong K_{2m}(S^2A) \cong K_0(S^2A) \cong K_1(SA) \\ &\cong K_0(A) \end{aligned}$$

and also

$$\begin{aligned} K_{2(m+1)+1} &= K_{2m+3}(A) \cong K_{2m+1}(S^2A) \cong K_1(S^2A) \cong K_0(SA) \\ &\cong K_1(A) . \end{aligned}$$

□

The proof of the Bott-periodicity used the Toeplitz extension. In the literature, the construction of the Bott-map is a common way of proving the Bott-periodicity.

Let A be a local C^*-algebra and $[p]_{00} - [p_n]_{00} \in K_0(A)$, with $p \in \mathrm{Proj}(M_k(A^+))$ for $k \geq n$, i.e. $p - p_n \in M_k(A)$ by theorem 3.4.40. Set

$$f_p(t) := e^{2i\pi t}p + \mathbb{1} - p ,$$

the it holds that

$$\begin{aligned} f_p(t)^* f_p(t) &= (e^{-2i\pi t}p + \mathbb{1} - p)(e^{2i\pi t}p + \mathbb{1} - p) \\ &= p^2 + e^{-2i\pi t}p - e^{-2i\pi t}p^2 + e^{2i\pi t}p + \mathbb{1} - p - e^{2i\pi t}p^2 - p + p^2 \\ &= p + e^{-2i\pi t}p - e^{-2i\pi t}p + e^{2i\pi t}p + \mathbb{1} - p - e^{2i\pi t}p - p + p \\ &= p + \mathbb{1} - p = \mathbb{1} = \ldots = f_p(t)f_p(t)^* , \end{aligned}$$

that is $f_p(t) \in U_k(A^+)$. Since $f_p(0) = \mathbb{1} = f_p(1)$, f_p is a loop in $U_k(A^+)$ with base point $\mathbb{1}$. The calculation

$$f_p(t)f_{p_n}(t)^{-1} = f_p(t)f_{p_n}(t)^* = (e^{2i\pi t}p + \mathbb{1} - p)(e^{-2i\pi t}p_n + \mathbb{1} - p_n)$$

$$\begin{aligned}\Rightarrow \quad f_p(0)f_{p_n}(0)^{-1} &= f_p(1)f_{p_n}(1)^{-1} = (p+\mathbb{1}-p)(p_n+\mathbb{1}-p_n)\\ &= pp_n + p - pp_n + p_n + \mathbb{1} - p_n - pp_n - p + pp_n\\ &= \mathbb{1} \in M_k(\mathbb{C})\end{aligned}$$

implies that $f_p f_{p_n}^{-1} \in U_k((SA)^+)$ (compare remark 3.5.41), allowing to define

$$\beta_A([p]_{00} - [p_n]_{00}) \coloneqq [f_p f_{p_n}^{-1}]_1 \in K_1(SA) \ .$$

Note that since f_p and f_{p_n} are loops in $U_k(A^+)$, so is $f_p f_{p_n}^{-1}$. Because of $p \equiv p_n \mod M_k(A)$, one can calculate that $f_p f_{p_n}^{-1} \equiv \mathbb{1} \mod U_k(SA)$, such that indeed $f_p f_{p_n}^{-1} \in U_k(SA)$, by definition 3.5.1.

Definition 3.7.8.
The map

$$\beta_A \colon K_0(A) \longrightarrow K_1(SA) \ , \quad \beta_A([p]_{00} - [p_n]_{00}) \coloneqq [f_p f_{p_n}^{-1}]_1$$

is called **Bott-map**.

We still need to show, that the Bott-map is well defined.

Lemma 3.7.9.
The Bott-map is a well defined group morphism. Furthermore, it is natural, i.e. for every $*$-morphism $\phi\colon A \to B$ of local C^*-algebras, the following diagram commutes:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\phi_*} & K_0(B) \\ \downarrow{\scriptstyle \beta_A} & & \downarrow{\scriptstyle \beta_B} \\ K_1(SA) & \xrightarrow{(S\phi)_*} & K_1(SB) \end{array}$$

Proof 3.7.10.
Let $p' \sim_h p$, then f_p is homotopic to $f_{p'}$ with base point $\mathbb{1}$. This shows that β_A is well defined.

We calculate with the help of lemma 3.5.32

$$\begin{aligned}&\beta_A([p]_{00} - [p_m]_{00} + [q]_{00} - [p_m]_{00})\\ &\quad = \beta_A\left([\mathrm{diag}(p,q)]_{00} - [\mathrm{diag}(p_n,p_m)]_{00}\right)\\ &\quad = \left[f_{\mathrm{diag}(p,q)} f_{\mathrm{diag}(p_n,p_m)}^{-1}\right]_1 = \left[\mathrm{diag}(f_p f_{p_n}^{-1})\mathrm{diag}(f_q f_{p_m}^{-1})\right]_1\\ &\quad = \left[\mathrm{diag}(f_p f_{p_n}^{-1}, f_q f_{p_m}^{-1})\right]_1 = [f_p f_{p_n}^{-1}]_1 [f_q f_{p_m}^{-1}]_1 \ ,\end{aligned}$$

which shows the morphism property.

For the commutativity of the diagram, let $[p]_{00} - [p_n]_{00} \in K_0(A)$, with $p \in \mathrm{Proj}(M_k(A^+))$ for $k \geq n$, i.e. $p - p_n \in M_k(A)$. Then

$$\beta_B(\phi_*([p]_{00} - [p_n]_{00})) = \beta_B([\phi(p)]_{00} - [\phi(p_n)]_{00})$$

$$= \beta_B([\phi(p)]_{00} - [p_n]_{00}) = [f_{\phi(p)} f_{p_n}^{-1}]_1$$
$$= [\phi(f_p) f_{p_n}^{-1}]_1$$

and with $\phi(f_{p_n}^{-1}) = \phi(f_{p_n}^*) = f_{p_n}^* = f_{p_n}^{-1}$:

$$(S\phi)_*(\beta_A([p]_{00} - [p_n]_{00})) = (S\phi)_*([f_p f_{p_n}^{-1}]_1) = [\phi(f_p f_{p_n}^{-1})]_1$$
$$= [\phi(f_p) f_{p_n}^{-1}]_1$$

□

Theorem 3.7.11.
The negative of the Bott-map $-\beta_A$ is inverse to the isomorphism $\partial\colon K_1(SA) \to K_0(A)$ form theorem 3.7.1.

Remark 3.7.12.
The Bott-map can be considered as a natural transformation, between the functors $K_1 \circ S$ and K_0, which the index $\beta_\bullet$ suggests.

Proof 3.7.13.
Consider the splitting short exact sequence (see proof 3.5.86):

$$0 \longrightarrow A \xrightarrow{\iota} A^+ \underset{j}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

It is enough to show the statement for unital A. Let $\partial\colon K_1 \circ S \to K_0(A \otimes \mathcal{K}(H^2))$ the index map of the reduced Toeplitz extension from proof 3.7.4, where it was shown, that ∂ is a natural isomorphism. Furthermore, let $\sigma\colon K_0(A) \to K_0(A \otimes \mathcal{K}(H^2))$ be the natural isomorphisma, induced by the $*$-morphisms

$$\xi\colon A \longrightarrow A \otimes \mathcal{K}(H^2) \subseteq \mathcal{T}_A\ , \quad a \longmapsto a \otimes e\ .$$

The induced morphisms $\sigma \equiv K_\bullet(\xi)$ are isomorphisms, because $K_\bullet(A \otimes \mathcal{K}(H^2)) \cong K_\bullet(A)$. The isomorphism constructed in proof 3.7.4, is just

$$\sigma^{-1} \circ \partial\colon K_1 \circ S \longrightarrow K_0\ .$$

∂ and σ are natural morphisms and $\sigma^{-1} \circ \partial$ is an isomorphism, so in order to prove that $\beta = (\sigma^{-1} \circ \partial)^{-1}$, it is enough to show that $-\sigma^{-1} \circ \partial \circ \beta = \mathrm{Id}$. This is an equation of natural transformations between the functors $K_1 \circ S$ and K_0.

With methods of category theory, it can be shown that such a natural transformation is given by its action on $A = \mathbb{C}$ and is well defined by its action on $[1]_0$. It holds that $\sigma_{\mathbb{C}}([1]_{00}) = [e]_{00}$ and by construction of the Bott-map $\beta_{\mathbb{C}}([1]_{00}) = [t \mapsto e^{2i\pi t}]_1 = [z]_1$, where $z \in \Omega\mathbb{C}$ is the identity on $\mathbb{S}^1$.

There is a lift for $\mathrm{diag}(z, \overline{z}) \in M_2(\mathcal{T})$, w.r.t. the map $\pi_{\mathbb{C}}\colon \mathcal{T}_{\mathbb{C}} = \mathcal{T} \to \Omega\mathbb{C}$, give by

$$v \coloneqq \begin{pmatrix} u & e \\ 0 & u^* \end{pmatrix}\ .$$

Indeed, $\pi_{\mathbb{C}}(u) = z$, so $\pi_{\mathbb{C}}(u^*) = \pi_{\mathbb{C}}(u)^* = z^* = \overline{z}$ and $\pi_{\mathbb{C}}(e) = \pi_{\mathbb{C}}(\mathbb{1} - uu^*) = 0$. It holds that

$$vv^* = \begin{pmatrix} u & e \\ 0 & u^* \end{pmatrix} \begin{pmatrix} u^* & 0 \\ e & u \end{pmatrix} = \begin{pmatrix} uu^* + e & u^*e \\ eu & u^*u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$v^*v = \begin{pmatrix} u^*u & u * e \\ eu & 1 + uu^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

This shows that v is unitary. Following the construction of the index map (definition 3.5.69):

$$\partial([z]_1) = [vp_1v^*]_{00} - [p_1]_{00} = \left[\begin{pmatrix} uu^* & 0 \\ 0 & 0 \end{pmatrix}\right]_{00} - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_{00} = \left[-\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right]_{00}$$

$$\Rightarrow \quad -\sigma_{\mathbb{C}}^{-1}(\partial([z]_1)) = -\sigma_{\mathbb{C}}^{-1}\left(\left[-\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right]_{00}\right) = \left[\begin{pmatrix} \sigma_{\mathbb{C}}^{-1}(e) & 0 \\ 0 & 0 \end{pmatrix}\right]_{00}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_{00} = [p_1]_{00} = [1]_{00} .$$

Yet, we have already $\beta_{\mathbb{C}}([1]_{00}) = [z]_1$, which shows $-\sigma^{-1} \circ \partial \circ \beta = \mathrm{Id}$. □

Theorem 3.7.14.
Let $J \subseteq A$ be a closed ideal, then there is an exact diagram of morphisms of abelian groups

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{j_*} & K_0(A) & \xrightarrow{\rho_*} & K_0(A/J) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\widetilde{\partial}} \\ K_1(A/J) & \xleftarrow[\rho_*]{} & K_1(A) & \xleftarrow[j_*]{} & K_1(J) \end{array}$$

where $\widetilde{\partial} := \partial \circ \beta_{A/J}$.

Proof 3.7.15.
First, we observe, that indeed $\beta_{A/J} \colon K_0(A/J) \to K_1(S(A/J)) \cong K_2(A/J)$ by definition 3.5.64 and $\partial \colon K_2(A/J) \to K_1(J)$, such that $\widetilde{\partial}$ is well defined. From theorem 3.5.75 the exactness follows in all terms but $K_0(A/J)$ and $K_1(J)$.

In the next step, we will use $\rho_* = K_0(\rho)$ to make the reasoning more transparent. Since β is a natural isomorphism (theorems 3.7.1 and 3.7.11), it holds that $\beta_A \circ K_0(\rho) = K_2(\rho) \circ \beta_{A/J} \colon K_0(A) \to K_2(A/J)$. But since $\mathrm{Im}(K_2(\rho)) = \mathrm{Ker}(\partial)$ for the index map, because of theorem 3.5.75, the sequence is exact in $K_0(A/J)$.

Exactness in $K_1(J)$ also follows from theorem 3.5.75, since $\mathrm{Im}(\partial) = \mathrm{Ker}(\iota_*)$. □

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Bibliography

[All17] A. Alldridge. *C*-Algebren und K-Theorie*. Ed. by A. Alldridge. 2017. URL: https://aalldridge.github.io/assets/lectures/2016-cstkty/cstkty.pdf.

[Bla86] B. Blackadar. *K-Theory for Operator Algebras*. 1st ed. .5. Mathematical Sciences Research Institute Publications. Springer-Verlag New York, 1986.

[Con97] J. B. Conway. *Conway, J. B.* Springer New York, 1997.

[Mur90] G.J. Murphy. *C^*-algebras and operator theory*. Academic Press, 1990.

[RLL00] M. Rørdam, F. Larsen, and N. Laustsen. *An Introduction to K-Theory for C*-Algebras*. London Mathematical Society Student Texts. Cambridge University Press, 2000.

[Weg93] N.E. Wegge-Olsen. *K-Theory and C*-algebras. A Friendly Approach*. Oxford University Press, 1993.

[Wer11] D. Werner. *Funktionalanalysis*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2011.

[Wik19] Wikipedia. *Splitting lemma*. 2019. URL: https://en.wikipedia.org/wiki/Splitting_lemma.